Using Spectral Analysis in the Study of Sarnak's Conjecture

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based on joint works with E.H. El Abdalaoui (Rouen)
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The Möbius function

$$\boldsymbol{\mu}(n) := \begin{cases} (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes } (k \geq 0), \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function $\boldsymbol{\mu}$ is *multiplicative:*

$$\boldsymbol{\mu}(n_1 n_2) = \boldsymbol{\mu}(n_1)\boldsymbol{\mu}(n_2)$$

whenever $\gcd(n_1, n_2) = 1$.

Sarnak's conjecture (2010)

For any topological dynamical system (X, T) with $h_{\mathsf{top}}(X, T) = 0$,
for any $f : X \to \mathbb{C}$ continuous,
for any $x \in X$,

$$\frac{1}{N} \sum_{1 \leq n \leq N} f(T^n x)\, \boldsymbol{\mu}(n) \xrightarrow[N \to \infty]{} 0.$$

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Spectral properties
Which properties of the Koopman operator $U_T : f \mapsto f \circ T$ on $L^2(m)$ imply the validity of Sarnak's conjecture for (X, T)?

Main tool: KBSZ criterion

Lemma (Katai, Bourgain-Sarnak-Ziegler)

Assume that (a_n) is a bounded sequence of complex numbers, such that

$$\limsup_{\substack{p,q\to\infty \\ \text{different primes}}} \left(\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\leq N} a_{pn}\overline{a}_{qn} \right| \right) = 0.$$

Then, for *any bounded multiplicative function $\boldsymbol{\nu}$*, we have

$$\frac{1}{N} \sum_{n\leq N} a_n \,\boldsymbol{\nu}(n) \xrightarrow[N\to\infty]{} 0.$$

Application to Sarnak's conjecture

For $a_n = f(T^n x)$: find sufficient conditions to have

$$\limsup_{\substack{p,q\to\infty \\ \text{different primes}}} \left(\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\leq N} f\big((T^p)^n x\big) \overline{f\big((T^q)^n x\big)} \right| \right) = 0.$$

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Indeed, $\frac{1}{N}\sum_{n\leq N}\mu(n) \xrightarrow[N\to\infty]{} 0.$

$\longrightarrow$ Correlation of orbits of T^p with orbits of T^q, for p, q different large primes.

Joinings and disjointness

Any limit of $\frac{1}{N_k}\sum_{n\leq N_k} f\big((T^p)^n x\big)\overline{f}\big((T^q)^n x\big)$ is of the form

$$\int_{X\times X} f(x_1)\overline{f}(x_2)\,d\kappa(x_1,x_2),$$

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$$J(T^p,T^q) := \{\text{joinings of } T^p \text{ and } T^q\}$$
$$J_e(T^p,T^q) := \{\text{ergodic joinings of } T^p \text{ and } T^q\}$$

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T^p and T^q are *disjoint* if $J(T^p,T^q) = \{m\otimes m\}$

Disjointness of prime powers

Theorem (Bourgain, Sarnak, Ziegler)

If for p, q different primes T^p and T^q are disjoint, then Sarnak's conjecture holds for (X, T).

Spectral disjointness

$f \in L^2_0(m)$. The *spectral measure* of f associated to the transformation T^p is the finite measure on the circle defined by

$$\widehat{\sigma_{f,T^p}(j)} := \langle f, U^j_{T^p} f \rangle_{L^2(m)}$$

T^p and T^q are *spectrally disjoint* if for each $f, g \in L^2_0(m)$, $\sigma_{f,T^p} \perp \sigma_{g,T^q}$.

Lemma

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Hence $F \perp G$ in $L^2(\kappa)$,
and $\kappa(A \times B) = m(A)m(B)$.

Corollary

If for p, q different primes T^p and T^q are *spectrally disjoint*, then Sarnak's conjecture holds for (X, T).

$\longrightarrow$ conditions for spectral disjointness of different prime powers?

Weak limits of powers (Ex. of Chacon)

Tower n

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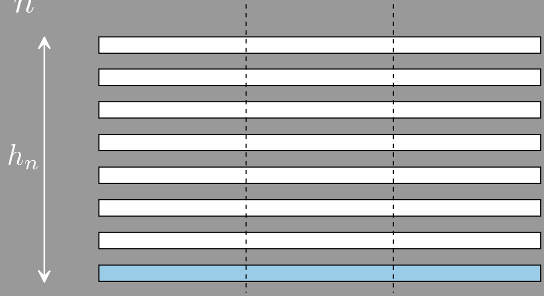

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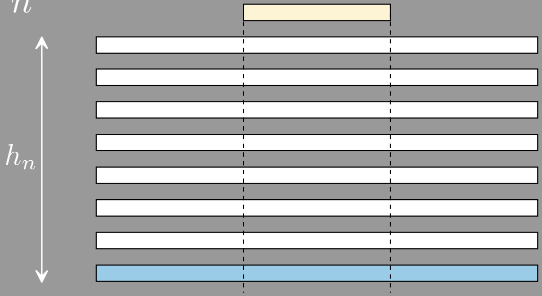

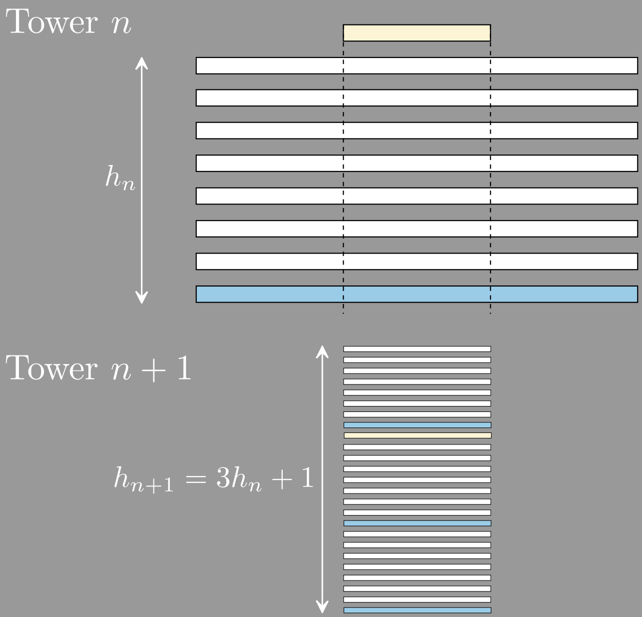
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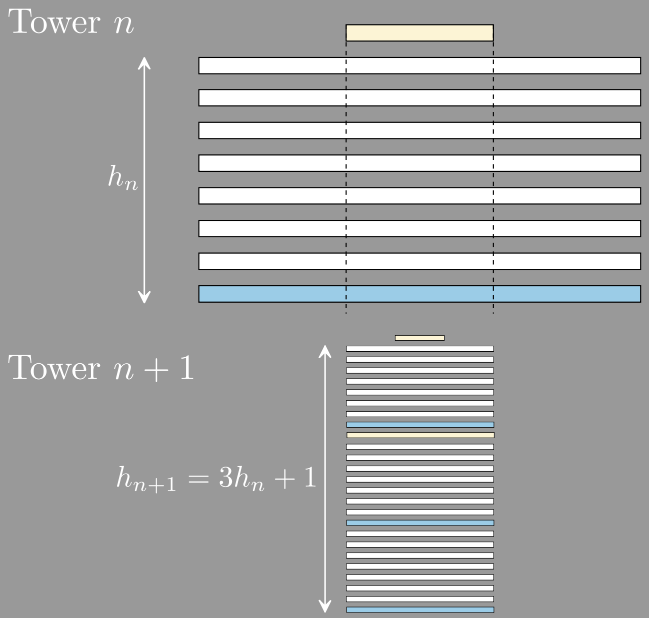
h_n
Tower $n+1$
$h_{n+1} = 3h_n + 1$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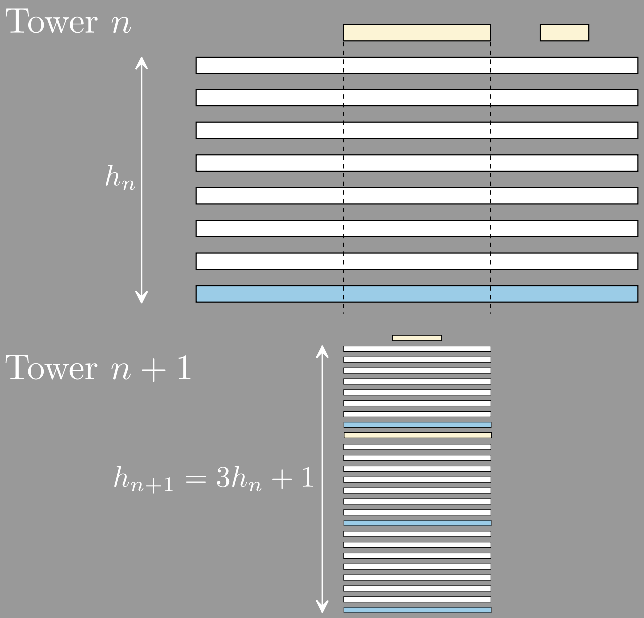
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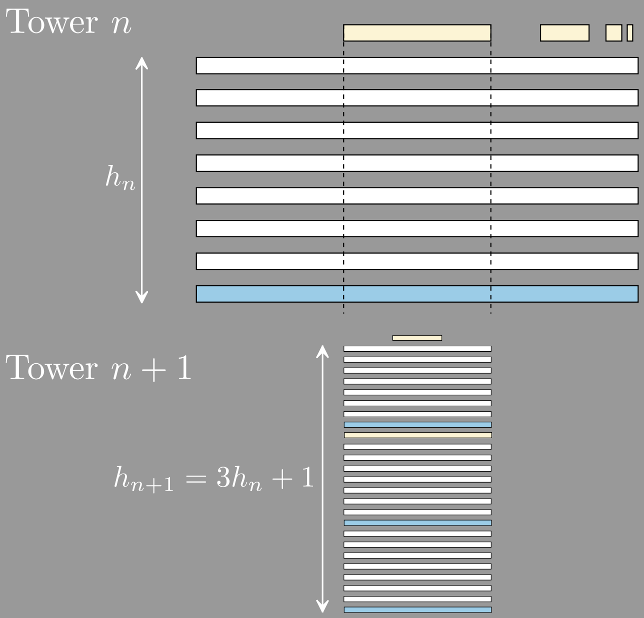
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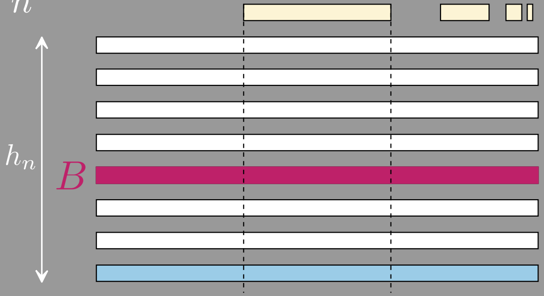

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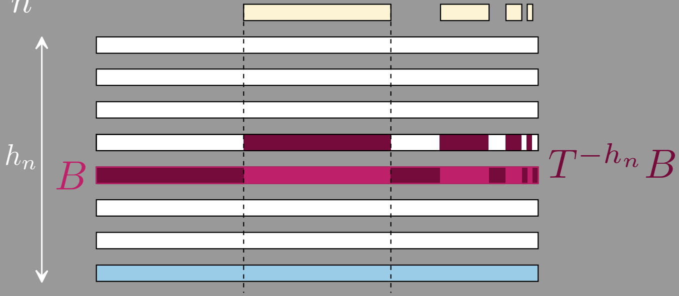
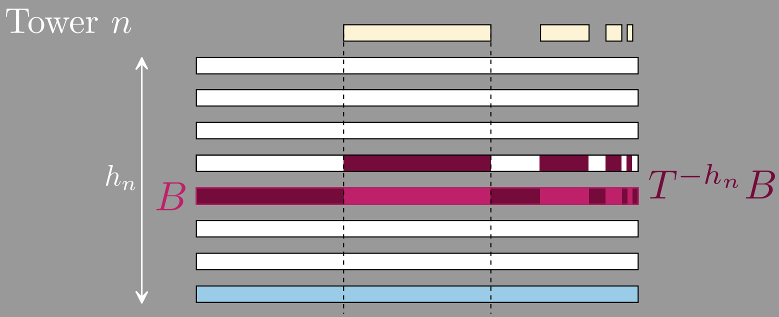


Tower n

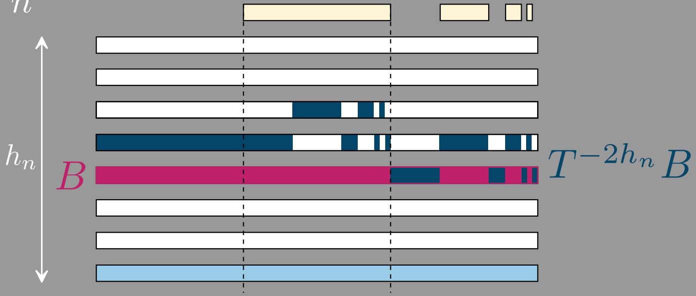
1/2 of $T^{-h_n}B$ is in B, 1/2 of $T^{-h_n}B$ is in TB

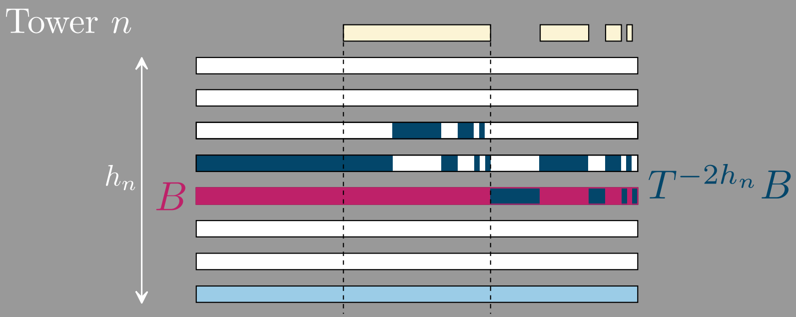


1/2 of $T^{-h_n}B$ is in B, 1/2 of $T^{-h_n}B$ is in TB

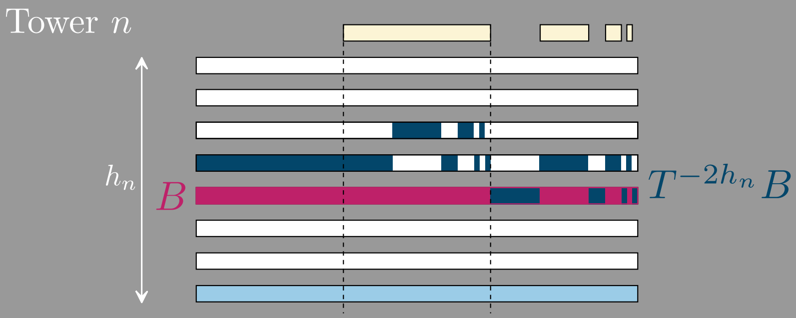
$$U_T^{-h_n} \xrightarrow[n\to\infty]{w} \frac{1}{2}(\mathsf{Id} + U_T)$$

Tower n





$1/6$ of $T^{-2h_n}B$ is in B, $1/6$ is in T^2B, $4/6$ is in TB.



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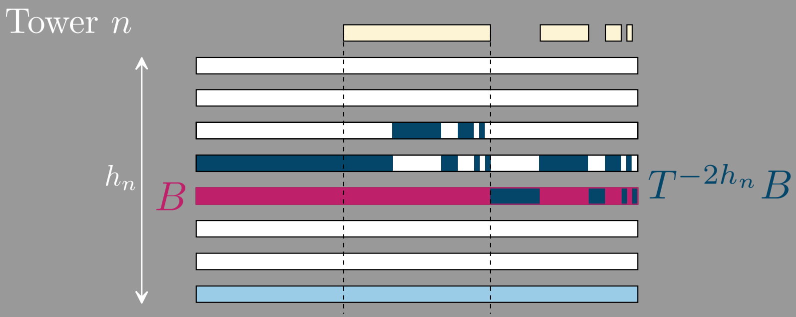

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In general,

$$U_T^{-kh_n} = U_{T^k}^{-h_n} \xrightarrow[n\to\infty]{w} P_k(U_T)$$

Spectral disjointness of T and T^2

If T and T^2 were not spectrally disjoint,
we could find

- $H_1 \subset L^2(X, m)$, stable by U_T
- $H_2 \subset L^2(X, m)$, stable by U_{T^2}
- a continuous measure σ on S^1

such that $(H_1, U_T) \approx (H_2, U_{T^2}) \approx \big(L^2(S^1, \sigma), \times z\big)$.

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Impossible!

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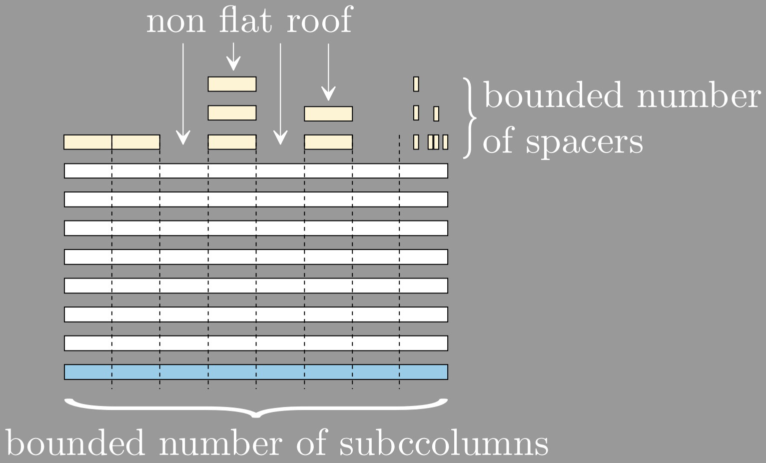
Spectral disjointness of powers

- For Chacon transformation, T^p and T^q are spectrally disjoint whenever $1 \leq p < q$.
- This result extends to a large class of rank-one transormations, including all weakly mixing constructions with *bounded parameters* and *non-flat* towers.


non flat roof
bounded number
of spacers
bounded number of subccolumns

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No hope to apply the method to non weakly-mixing systems:
If U_T has an eigenvalue $\alpha \neq 1$,

- T^p and T^q share a common eigenvalue α^{pq},
- T^p and T^q are never disjoint.

Applying KBSZ without disjointness?

In the case of disjoint powers, we have for $f \in L_0^2(m)$

$$\left(\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n \le N} f\big((T^p)^n x\big) \overline{f\big((T^q)^n x\big)} \right| \right) = 0.$$

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What we only need is

$$\limsup_{\substack{p,q\to\infty \\ \text{different primes}}} \left(\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\le N} f\big((T^p)^n x\big) \overline{f\big((T^q)^n x\big)} \right| \right) = 0.$$

AOP Property

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Definition

(X, m, T) has *Asymptotic Orthogonal Powers (AOP)* if $\forall f, g \in L^2_0(m)$,

$$\lim_{\substack{p,q\to\infty,\\ p,q \text{ different primes}}} \sup_{\kappa\in J_e(T^p,T^q)} \left| \int_{X\times X} f\otimes g\, d\kappa \right| = 0.$$

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By KBSZ criterion, any uniquely ergodic model of a system with AOP satisfies Sarnak's conjecture.

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Theorem

If (X, m, T) has discrete spectrum and is totally ergodic, then it has AOP.

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If (X, m, T) has discrete spectrum and is totally ergodic, then it has AOP.

$\longrightarrow$ includes examples where all powers are isomorphic.

$\longrightarrow$ extends to quasi-discrete spectrum systems, *e.g.*

$$T : (x_1, \ldots, x_d) \in \mathbb{T}^d \longmapsto (x_1+\alpha, x_2+x_1, \ldots, x_d+x_{d-1}).$$

Proof of AOP for discrete spectrum

$\forall f, g \in L_0^2(m)$,

$$\lim_{\substack{p,q\to\infty,\\ p,q \text{ different primes}}} \sup_{\kappa\in J_e(T^p,T^q)} \left| \int_{X\times X} f \otimes g \, d\kappa \right| = 0 \ ?$$

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$$\lim_{\substack{p,q\to\infty,\\ p,q \text{ different primes}}} \sup_{\kappa\in J_e(T^p,T^q)} \left| \int_{X\times X} f\otimes g\, d\kappa \right| = 0 \ ?$$

Enough to consider f and g eigenfunctions associated to irrational eigenvalues α and $\beta \in S^1$.

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But for α and β irrational eigenvalues, there exists at most one pair (p, q) such that $\alpha^p = \beta^q$. $\square$

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$\longrightarrow$ KBSZ criterion cannot be used when there exist rational eigenvalues.

Sarnak for discrete spectrum systems

Theorem (Huang, Wang, Zhang (2016))

Let (X, T) be a uniquely ergodic system with unique invariant measure m. If (X, m, T) has discrete spectrum, then Sarnak's conjecture holds for (X, T).

(even when there exist *rational* eigenvalues)

Sarnak for discrete spectrum systems

An essential argument in the proof: an estimation by Matomäki, Radziwill and Tao

$$\sup_{\alpha \in S^1} \frac{1}{N} \sum_{0 \le n < N} \left| \frac{1}{L} \sum_{0 \le \ell < L} \boldsymbol{\mu}(n+\ell)\alpha^{n+\ell} \right| \longrightarrow 0 \text{ as } N, L \to \infty,\ L \le N.$$