Using Spectral Analysis in the Study of Sarnak’s Conjecture

Thierry de la Rue
based on joint works with E.H. El Abdalaoui (Rouen) and Mariusz Lemańczyk (Toruń)

Laboratoire de Mathématiques Raphaël Salem
The Möbius function

\[ \mu(n) := \begin{cases} 
(-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes } (k \geq 0), \\
0 & \text{otherwise.}
\end{cases} \]

The Möbius function \( \mu \) is *multiplicative*:

\[ \mu(n_1 n_2) = \mu(n_1) \mu(n_2) \]

whenever \( \gcd(n_1, n_2) = 1 \).
Sarnak’s conjecture (2010)

For any topological dynamical system \((X, T)\) with \(h_{\text{top}}(X, T) = 0\),
for any \(f : X \rightarrow \mathbb{C}\) continuous,
for any \(x \in X\),

\[
\frac{1}{N} \sum_{1 \leq n \leq N} f(T^n x) \mu(n) \xrightarrow[N \to \infty]{} 0.
\]
Measurable dynamics point of view

Assume that $\left( X, T \right)$ is uniquely ergodic, with a unique invariant probability measure $m$. 
Assume that \((X, T)\) is uniquely ergodic, with a unique invariant probability measure \(m\).

Which properties of the measure preserving system \((X, m, T)\) imply the validity of Sarnak’s conjecture for \((X, T)\)?
Measurable dynamics point of view

Assume that \((X, T)\) is uniquely ergodic, with a unique invariant probability measure \(m\).

Which properties of the measure preserving system \((X, m, T)\) imply the validity of Sarnak’s conjecture for \((X, T)\)?

*Spectral properties*
Measurable dynamics point of view

Assume that \((X, T)\) is uniquely ergodic, with a unique invariant probability measure \(m\).

Which properties of the \(\textit{measure preserving system}\) \((X, m, T)\) imply the validity of Sarnak’s conjecture for \((X, T)\)?

\textit{Spectral properties}

Which properties of the Koopman operator \(U_T : f \mapsto f \circ T\) on \(L^2(m)\) imply the validity of Sarnak’s conjecture for \((X, T)\)?
Main tool: KBSZ criterion

Lemma (Katai, Bourgain-Sarnak-Ziegler)

Assume that \((a_n)\) is a bounded sequence of complex numbers, such that

\[
\limsup_{p,q \to \infty} \left( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} a_{pn} \bar{a}_{qn} \right| \right) = 0.
\]

Then, for any bounded multiplicative function \(\nu\), we have

\[
\frac{1}{N} \sum_{n \leq N} a_n \nu(n) \xrightarrow{N \to \infty} 0.
\]
Application to Sarnak’s conjecture

For $a_n = f(T^nx)$: find sufficient conditions to have

$$\limsup_{p,q \to \infty} \left( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} f((T^p)^n x) \overline{f((T^q)^n x)} \right| \right) = 0.$$
Application to Sarnak’s conjecture

For $a_n = f(T^n x)$: find sufficient conditions to have

$$\limsup_{p,q \to \infty} \left( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} f(T^p)^n x \frac{f(T^q)^n x}{f(T^q)^n x} \right| \right) = 0.$$  

For $\mu$, we can assume that $\int_X f \, dm = 0$. 

$\rightarrow$ For $\mu$, we can assume that $\int_X f \, dm = 0$. 

Application to Sarnak’s conjecture

For $a_n = f(T^n x)$: find sufficient conditions to have

$$\limsup_{p,q \to \infty} \left( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} f((T^p)^n x) f((T^q)^n x) \right| \right) = 0.$$  

$\longrightarrow$ For $\mu$, we can assume that $\int_X f \, dm = 0$  
Indeed, $\frac{1}{N} \sum_{n \leq N} \mu(n) \xrightarrow{N \to \infty} 0$.  

$\longrightarrow$ Correlation of orbits of $T^p$ with orbits of $T^q$, for $p,q$ different large primes.
Application to Sarnak’s conjecture

For $a_n = f(T^n x)$: find sufficient conditions to have

$$\limsup_{p,q \to \infty} \left( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} f((T^p)^n x) f((T^q)^n x) \right| \right) = 0.$$

---

For $\mu$, we can assume that $\int_X f \, dm = 0$

Indeed, $\frac{1}{N} \sum_{n \leq N} \mu(n) \xrightarrow{N \to \infty} 0$.

---

Correlation of orbits of $T^p$ with orbits of $T^q$, for $p, q$ different large primes.
Joinings and disjointness

Any limit of \( \frac{1}{N_k} \sum_{n \leq N_k} f((T^p)^n x) \overline{f((T^q)^n x)} \) is of the form

\[
\int_{X \times X} f(x_1) \overline{f(x_2)} \, d\kappa(x_1, x_2),
\]

where \( \kappa \) is a joining of \( T^p \) and \( T^q \),
Joinings and disjointness

Any limit of \( \frac{1}{N_k} \sum_{n \leq N_k} f((T^p)^n x) \overline{f}((T^q)^n x) \) is of the form

\[
\int_{X \times X} f(x_1) \overline{f}(x_2) \, d\kappa(x_1, x_2),
\]

where \( \kappa \) is a joining of \( T^p \) and \( T^q \), i.e. a \((T^p \times T^q)\)-invariant probability measure on \( X \times X \) with marginals \( m \).
Joinings and disjointness

Any limit of \( \frac{1}{N_k} \sum_{n \leq N_k} f((T^p)^n x) \overline{f((T^q)^n x)} \) is of the form

\[
\int_{X \times X} f(x_1) \overline{f(x_2)} \, d\kappa(x_1, x_2),
\]

where \( \kappa \) is a joining of \( T^p \) and \( T^q \), i.e. a \((T^p \times T^q)\)-invariant probability measure on \( X \times X \) with marginals \( m \).

\[
J(T^p, T^q) := \{ \text{joinings of } T^p \text{ and } T^q \}
\]
\[
J_e(T^p, T^q) := \{ \text{ergodic joinings of } T^p \text{ and } T^q \}\]
Joinings and disjointness

Any limit of \( \frac{1}{N_k} \sum_{n \leq N_k} f((T^p)^n x) \overline{f((T^q)^n x)} \) is of the form

\[
\int_{X \times X} f(x_1) \overline{f(x_2)} \, d\kappa(x_1, x_2),
\]

where \( \kappa \) is a joining of \( T^p \) and \( T^q \), i.e. a \((T^p \times T^q)\)-invariant probability measure on \( X \times X \) with marginals \( m \).

\[ J(T^p, T^q) := \{ \text{joinings of } T^p \text{ and } T^q \} \]
\[ J_e(T^p, T^q) := \{ \text{ergodic joinings of } T^p \text{ and } T^q \} \]

\( T^p \) and \( T^q \) are disjoint if \( J(T^p, T^q) = \{ m \otimes m \} \)
Disjointness of prime powers

**Theorem (Bourgain, Sarnak, Ziegler)**

If for $p, q$ different primes $T^p$ and $T^q$ are disjoint, then Sarnak’s conjecture holds for $(X, T)$. 
Spectral disjointness

\[ f \in L^2_0(m) \]. The spectral measure of \( f \) associated to the transformation \( T^p \) is the finite measure on the circle defined by

\[
\sigma_{f,T^p}(j) := \langle f, U_{T^p}^j f \rangle_{L^2(m)}
\]

\( T^p \) and \( T^q \) are spectrally disjoint if for each \( f, g \in L^2_0(m) \), \( \sigma_{f,T^p} \perp \sigma_{g,T^q} \).

**Lemma**

Spectral disjointness implies disjointness.
Spectral disjointness implies disjointness

let $\kappa$ be a joining of $T^p$ and $T^q$, and $A, B \subset X$
Spectral disjointness implies disjointness

Let $\kappa$ be a joining of $T^p$ and $T^q$, and $A, B \subset X$ in $(X \times X, T^p \times T^q, \kappa)$, set

$$F(x_1, x_2) := 1_A(x_1) - m(A),$$
$$G(x_1, x_2) := 1_B(x_2) - m(B).$$
Spectral disjointness implies disjointness

let $\kappa$ be a joining of $T^p$ and $T^q$, and $A, B \subset X$ in $(X \times X, T^p \times T^q, \kappa)$, set

$$F(x_1, x_2) := 1_A(x_1) - m(A),$$

$$G(x_1, x_2) := 1_B(x_2) - m(B).$$

Then

$$\sigma_{F,(T^p \times T^q)} = \sigma_{1_A - m(A), T^p}$$
Spectral disjointness implies disjointness

let $\kappa$ be a joining of $T^p$ and $T^q$, and $A, B \subset X$ in $(X \times X, T^p \times T^q, \kappa)$, set

\[
F(x_1, x_2) := 1_A(x_1) - m(A),
G(x_1, x_2) := 1_B(x_2) - m(B).
\]

Then

\[
\sigma_{F,(T^p \times T^q)} = \sigma_{1_A - m(A), T^p},
\]

\[
\sigma_{G,(T^p \times T^q)} = \sigma_{1_B - m(B), T^q}.
\]
Spectral disjointness implies disjointness

let $\kappa$ be a joining of $T^p$ and $T^q$, and $A, B \subset X$ in $(X \times X, T^p \times T^q, \kappa)$, set

$$F(x_1, x_2) := 1_A(x_1) - m(A),$$
$$G(x_1, x_2) := 1_B(x_2) - m(B).$$

Then

$$\sigma_{F,(T^p \times T^q)} = \sigma_{1_A - m(A), T^p},$$
$$\sigma_{G,(T^p \times T^q)} = \sigma_{1_B - m(B), T^q}. $$
Spectral disjointness implies disjointness

let $\kappa$ be a joining of $T^p$ and $T^q$, and $A, B \subset X$ in $(X \times X, T^p \times T^q, \kappa)$, set

$$F(x_1, x_2) := 1_A(x_1) - m(A),$$

$$G(x_1, x_2) := 1_B(x_2) - m(B).$$

Then

$$\sigma_{F,(T^p \times T^q)} \perp \sigma_{1_A - m(A), T^p}$$

and

$$\sigma_{G,(T^p \times T^q)} \perp \sigma_{1_B - m(B), T^q}.$$ 

Hence $F \perp G$ in $L^2(\kappa)$,
Spectral disjointness implies disjointness

let $\kappa$ be a joining of $T^p$ and $T^q$, and $A, B \subset X$ in $(X \times X, T^p \times T^q, \kappa)$, set

$$F(x_1, x_2) := 1_A(x_1) - m(A),$$
$$G(x_1, x_2) := 1_B(x_2) - m(B).$$

Then

$$\sigma_{F,(T^p \times T^q)} = \sigma_{1_A - m(A), T^p} \perp \sigma_{G,(T^p \times T^q)} = \sigma_{1_B - m(B), T^q}.$$ 

Hence $F \perp G$ in $L^2(\kappa)$, and $\kappa(A \times B) = m(A)m(B)$. 
Corollary

If for $p, q$ different primes $T^p$ and $T^q$ are \textit{spectrally disjoint}, then Sarnak’s conjecture holds for $(X, T)$.

→ conditions for spectral disjointness of different prime powers?
Weak limits of powers (Ex. of Chacon)
Tower $n$

$h_n$
Tower $n$

$h_n$
Tower $n$

$h_n$
Tower $n$

$h_n$

Tower $n+1$

$h_{n+1} = 3h_n + 1$
Tower $n$

$h_n$

Tower $n + 1$

$h_{n+1} = 3h_n + 1$
Tower $n$

$h_n$

Tower $n + 1$

$h_{n+1} = 3h_n + 1$
Tower $n$

$h_n$

Tower $n+1$

$h_{n+1} = 3h_n + 1$
Tower $n$

$h_n$

$B$

$T^{-h_n} B$
1/2 of $T^{-h_n} B$ is in $B$, 1/2 of $T^{-h_n} B$ is in $TB$
1/2 of $T^{-h_n}B$ is in $B$, 1/2 of $T^{-h_n}B$ is in $TB$

$$U_T^{-h_n} \xrightarrow{w} \frac{1}{2}(\text{Id} + U_T)$$
1/6 of $T^{-2h_n}B$ is in $B$, 1/6 is in $T^2B$, 4/6 is in $TB$. 
1/6 of $T^{-2h_n}B$ is in $B$, 1/6 is in $T^2B$, 4/6 is in $TB$.

$$U_T^{-2h_n} \xrightarrow{w/n \to \infty} \frac{1}{6}(\text{Id} + 4U_T + U_T^2)$$
1/6 of $T^{-2h_n}B$ is in $B$, 1/6 is in $T^2B$, 4/6 is in $TB$.

\[ U_{T}^{-2h_n} \xrightarrow{w} \frac{1}{6}(\text{Id} + 4U_T + U_T^2) \xrightarrow{n \to \infty} \]

In general,

\[ U_{T}^{-kh_n} = U_{T}^{-h_n} \xrightarrow{w} P_k(U_T) \xrightarrow{n \to \infty} \]
Spectral disjointness of $T$ and $T^2$

If $T$ and $T^2$ were not spectrally disjoint, we could find

- $H_1 \subset L^2(X, m)$, stable by $U_T$
- $H_2 \subset L^2(X, m)$, stable by $U_{T^2}$
- a continuous measure $\sigma$ on $S^1$

such that $(H_1, U_T) \approx (H_2, U_{T^2}) \approx (L^2(S^1, \sigma), \times z)$. 
Spectral disjointness of $T$ and $T^2$

If $T$ and $T^2$ were not spectrally disjoint, we could find

- $H_1 \subset L^2(X, m)$, stable by $U_T$
- $H_2 \subset L^2(X, m)$, stable by $U_T^2$ and by $U_T$
- a continuous measure $\sigma$ on $S^1$

such that $(H_1, U_T) \approx (H_2, U_T^2) \approx (L^2(S^1, \sigma), \times z)$. 
Spectral disjointness of $T$ and $T^2$

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$L^2(S^1, \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_T$</td>
<td>$U_{T^2}$</td>
<td>$\times z$</td>
</tr>
</tbody>
</table>

Hence $1 + \phi^2(z) = 1 + 4\phi(z) + \phi^2(z)$ ($\sigma$-a.e.)
Spectral disjointness of $T$ and $T^2$

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$L^2(S^1, \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_T$</td>
<td>$U_{T^2}$</td>
<td>$\times z$</td>
</tr>
<tr>
<td>$U_T$</td>
<td>$U_T$</td>
<td>$\times \phi(z)$, where $\phi^2(z) = z$</td>
</tr>
</tbody>
</table>


Hence $1 + \phi^2(z) = 1 + 4\phi(z) + \phi^2(z)$, $\sigma$-a.e.
Spectral disjointness of $T$ and $T^2$

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$L^2(S^1, \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_T$</td>
<td>$U_{T^2}$</td>
<td>$\times z$</td>
</tr>
<tr>
<td>$U_T$</td>
<td>$U_T$</td>
<td>$\times \phi(z)$, where $\phi^2(z) = z$</td>
</tr>
<tr>
<td>wlim $U_T^{-h_n}$</td>
<td>wlim $U_{T^2}^{-h_n}$</td>
<td>wlim $\times z^{-h_n}$</td>
</tr>
</tbody>
</table>
Spectral disjointness of $T$ and $T^2$

<table>
<thead>
<tr>
<th></th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$L^2(S^1, \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_T$</td>
<td>$U_{T^2}$</td>
<td>$\times \varphi(z)$, where $\varphi^2(z) = z$</td>
<td></td>
</tr>
<tr>
<td>$\text{wlim } U_T^{-h_n}$</td>
<td>$\text{wlim } U_{T^2}^{-h_n}$</td>
<td>$\text{wlim } \times z^{-h_n}$</td>
<td></td>
</tr>
<tr>
<td>$(\text{Id } + U_T)/2$</td>
<td></td>
<td>$\times (1 + z)/2$</td>
<td></td>
</tr>
</tbody>
</table>

$\text{wlim } U_T^{-h_n}$
## Spectral disjointness of $T$ and $T^2$

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$L^2(S^1, \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_T$</td>
<td>$U_{T^2}$</td>
<td>$\times z$</td>
</tr>
<tr>
<td>$U_T$</td>
<td>$U_T$</td>
<td>$\times \phi(z)$, where $\phi^2(z) = z$</td>
</tr>
<tr>
<td>$\text{wlim } U_T^{-h_n}$</td>
<td>$\text{wlim } U_{T^2}^{-h_n}$</td>
<td>$\text{wlim } \times z^{-h_n}$</td>
</tr>
<tr>
<td>$(\text{Id} + U_T)/2$</td>
<td></td>
<td>$\times (1 + z)/2$</td>
</tr>
<tr>
<td>$\frac{\text{Id} + 4U_T + U_T^2}{6}$</td>
<td></td>
<td>$\times \frac{1 + 4\phi(z) + \phi^2(z)}{6}$</td>
</tr>
</tbody>
</table>
Spectral disjointness of \( T \) and \( T^2 \)

<table>
<thead>
<tr>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>( L^2(S^1, \sigma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_T )</td>
<td>( U_{T^2} )</td>
<td>( \times z )</td>
</tr>
<tr>
<td>( U_T )</td>
<td>( U_T )</td>
<td>( \times \phi(z) ), where ( \phi^2(z) = z )</td>
</tr>
<tr>
<td>( \text{wlim } U_T^{-hn} )</td>
<td>( \text{wlim } U_{T^2}^{-hn} )</td>
<td>( \text{wlim } \times z^{-hn} )</td>
</tr>
<tr>
<td>( (\text{Id} + U_T)/2 )</td>
<td>( \frac{\text{Id} + 4U_T + U_T^2}{6} )</td>
<td>( \times \frac{1 + 4\phi(z) + \phi^2(z)}{6} )</td>
</tr>
</tbody>
</table>

Hence

\[
\frac{1 + \phi^2(z)}{2} = \frac{1 + 4\phi(z) + \phi^2(z)}{6} \quad (\sigma\text{-a.e.})
\]
Spectral disjointness of $T$ and $T^2$

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$L^2(S^1, \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_T$</td>
<td>$U_{T^2}$</td>
<td>$\times z$</td>
</tr>
<tr>
<td>$U_T$</td>
<td>$U_T$</td>
<td>$\times \phi(z)$, where $\phi^2(z) = z$</td>
</tr>
<tr>
<td>wlim $U_T^{-h_n}$</td>
<td>wlim $U_{T^2}^{-h_n}$</td>
<td>wlim $\times z^{-h_n}$</td>
</tr>
<tr>
<td>$(\text{Id} + U_T)/2$</td>
<td>$\frac{\text{Id} + 4U_T + U_{T^2}^2}{6}$</td>
<td>$\times \frac{1 + 4\phi(z) + \phi^2(z)}{6}$</td>
</tr>
</tbody>
</table>

Hence

$$\frac{1 + \phi^2(z)}{2} = \frac{1 + 4\phi(z) + \phi^2(z)}{6}$$

(\sigma\text{-a.e.})

Impossible!
Spectral disjointness of powers

- For Chacon transformation, $T^p$ and $T^q$ are spectrally disjoint whenever $1 \leq p < q$. 
Spectral disjointness of powers

- For Chacon transformation, $T^p$ and $T^q$ are spectrally disjoint whenever $1 \leq p < q$.
- This result extends to a large class of rank-one transformations, including all weakly mixing constructions with bounded parameters and non-flat towers.
non flat roof

bounded number of spacers

bounded number of subcolumns
Applying KBSZ without disjointness?

No hope to apply the method to non weakly-mixing systems:
Applying KBSZ without disjointness?

No hope to apply the method to non weakly-mixing systems:
If $U_T$ has an eigenvalue $\alpha \neq 1$, if $T_p$ and $T_q$ share a common eigenvalue $\alpha_{pq}$, $T_p$ and $T_q$ are never disjoint.
Applying KBSZ without disjointness?

No hope to apply the method to non weakly-mixing systems:
If $U_T$ has an eigenvalue $\alpha \neq 1$,
- $T^p$ and $T^q$ share a common eigenvalue $\alpha^{pq}$,
Applying KBSZ without disjointness?

No hope to apply the method to non weakly-mixing systems:
If $U_T$ has an eigenvalue $\alpha \neq 1$,
- $T^p$ and $T^q$ share a common eigenvalue $\alpha^{pq}$,
- $T^p$ and $T^q$ are never disjoint.
Applying KBSZ without disjointness?

In the case of disjoint powers, we have for \( f \in L^2_0(m) \)

\[
\left( \limsup_{N \to \infty} \frac{1}{N} \sum_{n \leq N} f\left((T^p)^n x\right) \overline{f\left((T^q)^n x\right)} \right) = 0.
\]
Applying KBSZ without disjointness?

In the case of disjoint powers, we have for \( f \in L^2_0(m) \)

\[
\left( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} f((T^p)^n x) f((T^q)^n x) \right| \right) = 0.
\]

What we only need is

\[
\limsup_{p,q \to \infty} \left( \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} f((T^p)^n x) f((T^q)^n x) \right| \right) = 0.
\]

\( \begin{align*}
\text{different primes}
\end{align*} \)
AOP Property

By KBSZ criterion, any uniquely ergodic model of a system with AOP satisfies Sarnak's conjecture.
AOP Property

**Definition**

\((X, m, T)\) has **Asymptotic Orthogonal Powers (AOP)** if \(\forall f, g \in L_0^2(m)\),

\[
\lim_{p,q \to \infty, \ p,q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0.
\]
AOP Property

**Definition**

\((X, m, T)\) has *Asymptotic Orthogonal Powers (AOP)* if \(\forall f, g \in L^2_0(m),\)

\[
\lim_{p, q \to \infty, \\text{ } p, q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0.
\]

By KBSZ criterion, any uniquely ergodic model of a system with AOP satisfies Sarnak’s conjecture.
AOP for (quasi-)discrete spectrum

**Theorem**

If \((X, m, T)\) has discrete spectrum and is totally ergodic, then it has AOP.
AOP for (quasi-)discrete spectrum

**Theorem**

If \((X, m, T)\) has discrete spectrum and is totally ergodic, then it has AOP.

→ includes examples where all powers are isomorphic.
AOP for (quasi-)discrete spectrum

Theorem

If \((X, m, T)\) has discrete spectrum and is totally ergodic, then it has AOP.

\[ \rightarrow \text{ includes examples where all powers are isomorphic.} \]

\[ \rightarrow \text{ extends to quasi-discrete spectrum systems, e.g.} \]

\[ T : (x_1, \ldots, x_d) \in \mathbb{T}^d \mapsto (x_1 + \alpha, x_2 + x_1, \ldots, x_d + x_{d-1}). \]
Proof of AOP for discrete spectrum

\[ \forall f, g \in L_0^2(m), \]

\[ \lim_{p,q \to \infty, \ p,q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0? \]
Proof of AOP for discrete spectrum

\[ \forall f, g \in L^2_0(m), \]

\[ \lim_{p,q \to \infty, p,q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0 \ ? \]

Enough to consider \( f \) and \( g \) eigenfunctions associated to irrational eigenvalues \( \alpha \) and \( \beta \in S^1 \).
Proof of AOP for discrete spectrum

\[ \forall f, g \in L^2_0(m), \]

\[
\lim_{p,q \to \infty, \ p,q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0 ?
\]

Enough to consider \( f \) and \( g \) eigenfunctions associated to irrational eigenvalues \( \alpha \) and \( \beta \in S^1 \).

For \( \kappa \in J_e(T^p, T^q) \), in \( (X \times X, T^p \times T^q) \)}
Proof of AOP for discrete spectrum

\[ \forall f, g \in L^2_0(m), \]

\[ \lim_{p,q \to \infty, \ p,q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0 ? \]

Enough to consider \( f \) and \( g \) eigenfunctions associated to irrational eigenvalues \( \alpha \) and \( \beta \in S^1 \).

For \( \kappa \in J_e(T^p, T^q) \), in \( (X \times X, T^p \times T^q, \kappa) \)

\[ f \otimes 1 \text{ is an eigenfunction associated to } \alpha^p \]
Proof of AOP for discrete spectrum

\[ \forall f, g \in L^2_0(m), \]

\[
\lim_{p,q \to \infty, \ p,q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0 ?
\]

Enough to consider \( f \) and \( g \) eigenfunctions associated to irrational eigenvalues \( \alpha \) and \( \beta \in S^1 \). For \( \kappa \in J_e(T^p, T^q) \), in \((X \times X, T^p \times T^q, \kappa)\)

\[ \triangleright f \otimes 1 \text{ is an eigenfunction associated to } \alpha^p \]

\[ \triangleright 1 \otimes g \text{ is an eigenfunction associated to } \beta^q \]
Proof of AOP for discrete spectrum

\[ \forall f, g \in L^2_0(m), \]

\[ \lim_{p,q \to \infty, \ p,q \text{ different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0 ? \]

Enough to consider \( f \) and \( g \) eigenfunctions associated to irrational eigenvalues \( \alpha \) and \( \beta \in S^1 \).

For \( \kappa \in J_e(T^p, T^q) \), in \((X \times X, T^p \times T^q, \kappa)\):

\begin{itemize}
  \item \( f \otimes 1 \) is an eigenfunction associated to \( \alpha^p \)
  \item \( 1 \otimes g \) is an eigenfunction associated to \( \beta^q \)
  \item if \( \alpha^p \neq \beta^q \), then \( f \otimes 1 \perp 1 \otimes g \)
\end{itemize}
Proof of AOP for discrete spectrum

\[ \forall f, g \in L^2_0(m), \]

\[
\lim_{p,q \to \infty, \\text{\(p,q\) different primes}} \sup_{\kappa \in J_e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0 ?
\]

Enough to consider \( f \) and \( g \) eigenfunctions associated to irrational eigenvalues \( \alpha \) and \( \beta \in S^1 \).

For \( \kappa \in J_e(T^p, T^q) \), in \((X \times X, T^p \times T^q, \kappa)\)

\[ \begin{align*}
\text{\(\ôtimes\)} f \otimes 1 \text{ is an eigenfunction associated to } \alpha^p \\
\text{\(\ôtimes\)} 1 \otimes g \text{ is an eigenfunction associated to } \beta^q \\
\text{\(\ôtimes\)} \text{ if } \alpha^p \neq \beta^q, \text{ then } f \otimes 1 \perp 1 \otimes g
\end{align*} \]

But for \( \alpha \) and \( \beta \) irrational eigenvalues, there exists at most one pair \( (p, q) \) such that \( \alpha^p = \beta^q \).  \[\square\]
The case of rational eigenvalues
The case of rational eigenvalues

- KBSZ criterion ensures orthogonality to any bounded multiplicative function
The case of rational eigenvalues

- KBSZ criterion ensures orthogonality to any bounded multiplicative function
- Existence of periodic multiplicative functions (Dirichlet characters)
The case of rational eigenvalues

- KBSZ criterion ensures orthogonality to *any bounded multiplicative function*
- Existence of *periodic* multiplicative functions (Dirichlet characters)
- Those functions are output by rotations on a finite number of points
The case of rational eigenvalues

- KBSZ criterion ensures orthogonality to any bounded multiplicative function
- Existence of periodic multiplicative functions (Dirichlet characters)
- Those functions are output by rotations on a finite number of points

→ KBSZ criterion cannot be used when there exist rational eigenvalues.
Theorem (Huang, Wang, Zhang (2016))

Let \((X, T)\) be a uniquely ergodic system with unique invariant measure \(m\). If \((X, m, T)\) has discrete spectrum, then Sarnak’s conjecture holds for \((X, T)\).

(even when there exist \textit{rational} eigenvalues)
Sarnak for discrete spectrum systems

An essential argument in the proof: an estimation by Matomäki, Radziwill and Tao

$$\sup_{\alpha \in S^1} \frac{1}{N} \sum_{0 \leq n < N} \left| \frac{1}{L} \sum_{0 \leq \ell < L} \mu(n + \ell) \alpha^{n+\ell} \right| \to 0 \text{ as } N, L \to \infty, \; L \leq N.$$