Balance and eigenvalues for S-adic subshifts

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Balance on factors

Let (X, T) be a minimal subshift

- Let $\mathcal{L}_X \subset \mathcal{A}^*$ be its language (set of factors)
- Let $|x|_v$ stands for the number of occurrences of the finite word v in the finite word x

The minimal subshift (X, T) is balanced on $v \in \mathcal{L}_X$ if there exists $C > 0$ such that for any pair (x, y) of factors of the same length in $\mathcal{L}_X$

$$||x|_v - |y|_v| \leq C$$

It is balanced on factors if it is balanced on all $v \in \mathcal{L}_X$

Frequencies and symbolic discrepancy

Let u be an infinite word in $\mathcal{A}^{\mathbb{N}}$

- The frequency μ_v of a finite word $v \in \mathcal{A}^*$ is defined as

$$\mu_v = \lim_{n \to +\infty} \frac{|u_0 \cdots u_{N-1}|_v}{N}$$

- One has uniform frequency for v if the convergence of

$$\frac{|u_k \cdots u_{k+N-1}|_v}{N}$$

toward μ_v is uniform in k when N tends to infinity

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- Symbolic discrepancy
$$\Delta_u(v, N) = ||u_0 u_1 \ldots u_{N-1}|_v - N \cdot \mu_v|$$
$$\Delta_X(v, N) = \sup_{w \in \mathcal{L}_X,\ |w|=N} ||w|_v - N \cdot \mu_v|$$
- The minimal subshift (X, T) has uniform frequency for v if $\Delta_X(v, N)$ is in $o(N)$

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Example Let X_σ be the Fibonacci shift generated by $\sigma : 0 \mapsto 01$, $1 \mapsto 0$. For any v, $\Delta_{X_\sigma}(v, N)$ is bounded. X_σ is balanced.

Balance and equidistribution

The minimal subshift (X, T) is balanced on the factor v iff there exist $C > 0$ and μ_v such that for any factor $w \in \mathcal{L}_X$

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Proof
Assume that there exist $C > 0$ and μ_v such that $||w|_v - \mu_v|w|| \leq C$ for every factor $w \in \mathcal{L}_X$
For every pair of factors w_1 and w_2 with the same length n

$$||w_1|_v - |w_2|_v| \leq ||w_1|_v - n\mu_v| + ||w_2|_v - n\mu_v| \leq 2C$$

Hence X is $2C$-balanced on v

Balancedness implies the existence of uniform letter frequencies

Proof Assume that X is C-balanced on v

Let N_p be such that for every factor of length p of X, the number of occurrences of v belongs to the set

$$\{N_p, N+1, \cdots, N_p + C\}$$

The sequence $(N_p/p)_{p\in\mathbb{N}}$ is a Cauchy sequence. Indeed consider a factor w of length pq

$$pN_q \leq |w|_v \leq pN_q + pC, \qquad qN_p \leq |w|_v \leq qN_p + qC$$

$$-C/p \leq N_p/p - N_q/q \leq C/q$$

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Let $\mu_v = \lim N_q/q$

$$-C \leq N_p - p\mu_v \leq 0 \quad (q \to \infty)$$

Then, for any factor w

$$||w|_v - \mu_v|w|| \leq C$$

Balancedness and coboundaries

(X, T) is balanced on v iff the ergodic sums for $f_v = \mathbf{1}_{[v]} - \mu_v$ are bounded

$$\sum_{n=0}^{N-1} f_v(T^n(u)) = |u_0 \cdots u_{N+|v|-1}|_v - \mu_v N$$

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Theorem [Gottschalk-Hedlund] Let X be a compact metric space and $T\colon X \to X$ be a minimal homeomorphism. Let $f\colon X \to \mathbb{R}$ be a continuous function. Then f is a coboundary

$$f = g - g \circ T$$

for a continuous function g if and only if there exists x and there exists $C > 0$ such that for all N

$$|\sum_{n=0}^{N} f(T^n(x))| < C$$

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Balancedness and topological eigenvalues

(X, T) is balanced on v iff $f_v = \mathbf{1}_{[v]} - \mu_v$ is a coboundary

$$\text{Take} \quad f = \mathbf{1}_{[v]} - \mu_v \quad \rightsquigarrow \quad f = g - g \circ T$$

$$\exp^{2i\pi g \circ T} = \exp^{2i\pi \mu_v} \exp^{2i\pi g}$$

$\exp^{2i\pi g}$ is a continuous eigenfunction associated with the eigenvalue $\exp^{2i\pi \mu_v} \rightsquigarrow$ Topological rotation factor

If (X, T) is balanced on v, then μ_v is an additive topological eigenvalue

Outline

- Balancedness: from letters to factors
- Topological vs. measure-theoretical eigenvalues
- Balancedness for S-adic words
  - Pisot case
  - Dendric subshifts (cf Paulina's lecture)

Two-letter factor substitution

Given a substitution σ, consider the finite set $\mathcal{L}_2(X_\sigma)$ as an alphabet and define the two-letter factor substitution σ_2 on $\mathcal{L}_2(X_\sigma)$ as follows

for every $u = ab \in \mathcal{L}_2(X_\sigma)$, $\sigma_2(u)$ is the word over $\mathcal{L}_2(X_\sigma)$ made of the first $|\sigma(a)|$ factors of length 2 in $\sigma(u)$

For instance, if $ab \in \mathcal{L}_2(X)$ with $\sigma(a) = a_0 \cdots a_r$, $\sigma(b) = b_0 \cdots b_s$, then

$$\sigma_2(ab) = (a_0 a_1)(a_1 a_2) \cdots (a_{r-1} a_r)(a_r b_0)$$

- If the substitution σ is primitive, then σ_2 is also primitive, and σ_2 has the same Perron–Frobenius eigenvalue as σ
- Frequencies of factors are provided by the renormalized Perron–Frobenius eigenvector of M_{σ_2}

[Queffélec]

Balancedness and substitutions

Let σ be a primitive substitution.

Theorem [Adamczewski]

- If σ (resp. σ_2) is a Pisot substitution, then the subshift X_σ is balanced on letters (resp. on factors).
- Conversely, if X_σ is balanced on letters (resp. on factors), then the Perron–Frobenius eigenvalue of M_σ (resp. M_{σ_2}) is the unique eigenvalue of M_σ (resp. M_{σ_2}) that is larger than 1 in modulus, and all possible eigenvalues of modulus one of M_σ (resp. M_{σ_2}) are roots of unity.

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Example [Cassaigne-Pytheas Fogg-Minervino]

$\sigma\colon 1 \mapsto 121, 2 \mapsto 32, 3 \mapsto 321$. The eigenvalues of its substitution matrix are $\{1, \frac{3\pm\sqrt{5}}{2}\}$ and it is balanced on factors.

Proof Consider the Sturmian substitution $\tau\colon 3 \mapsto 30, 0 \mapsto 300$. The subshift (X_σ, T) is deduced from the Sturmian shift (X_τ, T) by applying the substitution $\varphi\colon 0 \mapsto 21, 3 \mapsto 3$, which preserves balancedness.

Balancedness: letters vs. factors

Consider the Thue–Morse substitution $\sigma\colon 0 \mapsto 01,\ 1 \mapsto 10$

- One has $\mathcal{L}_2(\sigma) = \{00, 01, 10, 11\}$
- One has $\sigma(00) = 0101$ and $\sigma_2(00) = (01)(10)$
- One checks that $\sigma^{(2)}(a) \mapsto bc,\ b \mapsto bd,\ c \mapsto ca,\ d \mapsto cb$, by setting $a = 00,\ b = 01,\ c = 10,\ d = 11$
- The eigenvalues of M_σ are 2 and 0, and the eigenvalues of M_{σ_2} are 0, 1, -1 and 2.
- The subshift (X_σ, T) is balanced on letters but it is unbalanced on any factor of length ℓ, with $\ell \geq 2$

How to detect imbalances for rational frequencies

- KR towers $\mathcal{P}_n = \{T^j\sigma^n([ab]) : ab \in \mathcal{L}_2(X), 0 \leq j < |\sigma^n(a)|\}$
- If $f \in C(X, \mathbb{Z})$ is a coboundary, then it is the coboundary of some $h \in C(X, \mathbb{Z}) \rightsquigarrow$ locally constant

$T \uparrow$	$\subseteq [101]$	$[110]$	$[001]$	$\subseteq [010]$
	$\subseteq [0101]$	$[0110]$	$[1001]$	$\subseteq [1010]$
	00	**01**	**10**	**11**

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- Let $f \in C(X_\sigma, \mathbb{Z})$ and let $\phi_n \in \mathbb{R}^{\mathcal{L}_2(X_\sigma)}$

$$\phi_n(ab) = \sum_{j=0}^{|\sigma^n(a)|-1} f\,|_{T^j\sigma^n([ab])} \quad \forall ab \in \mathcal{L}_2(X_\sigma)$$

- If f is a coboundary, then $\phi_n \in \beta(R_1(X_\sigma))$ for n large enough [Host, Durand-Host-Perrin] $\rightsquigarrow$ $\phi_n(aa) = 0$

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- If f is a coboundary, then $\phi_n \in \beta(R_1(X_\sigma))$ for n large enough [Host, Durand-Host-Perrin] $\rightsquigarrow$ $\phi_n(aa) = 0$
- The frequency μ_v of a factor v is of the form $\mu_v = p_v/q_v$ with $p_v = 1$, and $q_v \in \{3 \cdot 2^{m+1}, 3 \cdot 2^m\}$ [Dekking]
- Take $f = \mathbf{1}_v - \mu_v$
- $\phi_n(aa) = \alpha_{aa}\left(1 - \frac{p_v}{q_v}\right) - (|\sigma^n(a)| - \alpha_{aa}) \cdot \frac{p_v}{q_v}$
  α_{aa} is the number of levels in the $aa-$tower in which all elements begin with the word v
- $q_v\alpha_{aa} = p_v\sigma^n(a) \rightsquigarrow$ Contradiction! ☺ [B.-Cecchi]

Pisot S-adic shifts

Pisot substitutions

Pisot substitution σ is primitive and its Perron–Frobenius eigenvalue is a Pisot number

Fact Symbolic dynamical systems generated by Pisot substitutions are balanced

Pisot irreducible substitution The characteristic polynomial of its incidence matrix is irreducible

The Pisot substitution conjecture

Let σ be a Pisot irreducible substitution

$$\begin{tikzcd} X_\sigma \arrow[r, "T \text{ shift}"] \arrow[d] & X_\sigma \arrow[d] \\ G \arrow[r, "g \mapsto ag"'] & G \end{tikzcd}$$

- (X_σ, T) is measure-theoretically isomorphic to a translation on a compact abelian group
- (X_σ, T) has pure discrete spectrum

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Substitutive structure + Algebraic assumption (Pisot)

= Order (discrete spectrum)

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The Pisot substitution Conjecture dates back to the 80's

[Bombieri-Taylor, Rauzy, Thurston]

The conjecture is proved for two-letter alphabets

[Host, Barge-Diamond, Hollander-Solomyak]

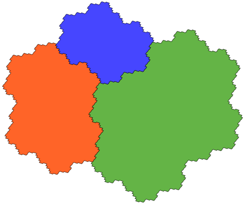
Theorem [Barge] If σ is a Pisot irreducible substitution that is injective on initial letters, and constant on final letters, then (X_σ, T) has pure discrete spectrum

Tribonacci dynamics and Tribonacci Kronecker map

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Theorem [Rauzy'82] The symbolic dynamical system (X_σ, T) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$



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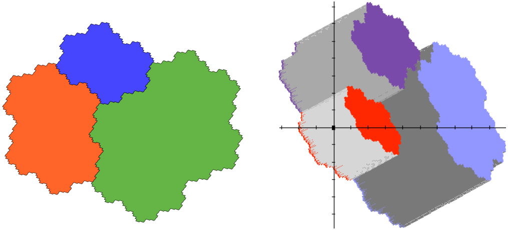

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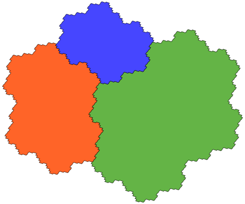

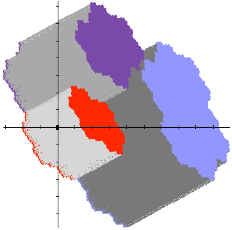

Markov partition for the toral automorphism $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

S-adic expansions and non-stationary dynamics

Definition An infinite word u is said S-adic if there exist

- a set of substitutions $\mathcal{S}$
- an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

$$u = \lim_{n\to+\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

The terminology comes from Vershik adic transformations
Bratteli diagrams

S stands for substitution, adic for the inverse limit

We consider a multidimensional continued fraction algorithm that governs the substitutions

Dictionary

S-adic description of a minimal symbolic dynamical system $\rightleftharpoons$ multidimensional continued fraction algorithm that governs its letter frequency vector/ invariant measure

- S-adic expansion
- Unique ergodicity
- Linear recurrence
- Balance and Pisot properties
- Two-sided sequences of substitutions
- Shift on sequences of substitutions

- Continued fraction
- Convergence
- Bounded partial quotients
- Strong convergence
- Natural extension
- Continued fraction map

Which continued fraction algorithms?

We focus here on two algorithms

- Arnoux-Rauzy algorithm

$$(a, b, c) \mapsto (a - (b + c), b, c) \text{ if } a \geq b + c$$

- Brun algorithm

$$(a, b, c) \mapsto \text{Sort}(a, b, c - b) \text{ if } a \leq b \leq c$$

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- Arnoux-Rauzy algorithm
  - Defined on a set of zero measure
  - Coding plus projection of an exchange of 6 intervals on the circle
  - They code particular systems of isometries (thin type) (pseudogroups of rotations) [Arnoux-Yoccoz, Novikov, De Leo-Dynnikov, Gaboriau-Levitt-Paulin, etc.]
  - A geometric context: natural suspension flow
  - Invariant measure, simplicity of the Lyapunov exponent [Avila-Hubert-Skripchenko]

- Brun algorithm
  - Invariant measure, natural extension, Lyapunov exponents, exponential convergence are well-known

S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0,1]^2$, the translation by (α, β) on the torus $\mathbb{T}^2$ admits a natural symbolic coding provided by the S-adic system associated with Brun multidimensional continued fraction algorithm applied to (α, β)

- For almost every Arnoux-Rauzy word, the associated S-adic system has pure discrete spectrum

Arnoux-Rauzy words

$$\sigma_1: \begin{array}{ll} 1 & \mapsto 1 \\ 2 & \mapsto 21 \\ 3 & \mapsto 31 \end{array} \qquad \sigma_2: \begin{array}{ll} 1 & \mapsto 12 \\ 2 & \mapsto 2 \\ 3 & \mapsto 32 \end{array} \qquad \sigma_3: \begin{array}{ll} 1 & \mapsto 13 \\ 2 & \mapsto 23 \\ 3 & \mapsto 3 \end{array}$$

$$u = \lim_{n\to\infty} \sigma_{i_0}\sigma_{i_1}\cdots\sigma_{i_n}(1)$$

and every letter in $\{1, 2, 3\}$ occurs infinitely often in $(i_n)_{n\geq 0}$

Example The Tribonacci substitution and its fixed point

• The set of the letter density vectors of AR words has zero measure [Arnoux-Starosta] and even Hausdorff dimension < 2 [Avila-Hubert-Skripchenko]

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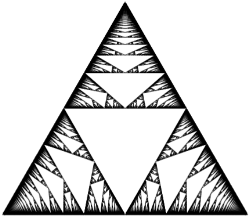

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- The set of the letter density vectors of AR words has zero measure [Arnoux-Starosta] and even Hausdorff dimension < 2 [Avila-Hubert-Skripchenko]
- There exist AR words that are not balanced [Cassaigne-Ferenczi-Zamboni]
- There exist AR words that are (measure-theoretically) weak mixing [Cassaigne-Ferenczi-Messaoudi]

Example

Let $(i_n) \in \{1,2,3\}^{\mathbb{N}}$ be the fixed point of Tribonacci substitution

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

$$(i_n) = \sigma^{\infty}(1) = 12131211213121312112 13$$

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$$\text{Take} \quad u = \lim_{n \to \infty} \sigma_{i_0} \sigma_{i_1} \cdots \sigma_{i_n}(1)$$

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Theorem [B-Steiner-Thuswaldner] (X_u, T) has pure discrete spectrum

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Theorem [B-Steiner-Thuswaldner] (X_u, T) has pure discrete spectrum for any Arnoux-Rauzy word u whose directive sequence (i_n) belongs to the shift generated by a primitive substitution

S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0, 1]^2$, the S-adic system provided by the Brun multidimensional continued fraction algorithm applied to (α, β) is measurably conjugate to the translation by (α, β) on the torus $\mathbb{T}^2$
- For almost every Arnoux-Rauzy word, the associated S-adic system has discrete spectrum

Proof Based on

- "adic IFS" (Iterated Function System)
- Theorem [Avila-Delecroix]
  - The Arnoux-Rauzy S-adic system is Pisot
- Theorem [Avila-Hubert-Skripchenko]
  - A measure of maximal entropy for the Rauzy gasket
- Finite products of Brun/Arnoux-Rauzy substitutions have discrete spectrum [B.-Bourdon-Jolivet-Siegel] Finiteness property

S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0,1]^2$, the S-adic system provided by the Brun multidimensional continued fraction algorithm applied to (α, β) is measurably conjugate to the translation by (α, β) on the torus $\mathbb{T}^2$
- For almost every Arnoux-Rauzy word, the associated S-adic system has discrete spectrum

S-adic Pisot conjecture Every unimodular and algebraically irreducible S-adic Pisot system has pure discrete spectrum

Pisot S-adic systems

- Let $\mathcal{S}$ be a set of unimodular substitutions
- Let (D, Σ, ν) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an ergodic subshift equipped with a probability measure ν. We assume log-integrability
- We consider the generic behaviour of the cocyle $A_n(\boldsymbol{\sigma}) = M_{\sigma_0} \cdots M_{\sigma_n}$ for $\boldsymbol{\sigma} = (\sigma_n) \in D$

The S-adic system (D, S, ν) is said to Pisot S-adic if the Lyapunov exponents $\theta_1, \theta_2, \ldots, \theta_d$ of (D, Σ, ν) satisfy

$$\theta_1 > 0 > \theta_2 \geq \theta_3 \geq \cdots \geq \theta_d$$

The PRICE to pay

$$M_{[k,\ell]} = M_k \cdots M_{\ell-1} \qquad u^{(k)} = \lim_{n\to\infty} \sigma_{i_k}\sigma_{i_1}\cdots\sigma_{i_n}(a) \quad \rightsquigarrow (X^{(k)}, T)$$

- (P) Primitivity $\forall k$, $M_{[k,\ell)} > 0$ for some $\ell > k$
- (R) Recurrence For each ℓ there exist $n = n(\ell)$ s.t.

$$(\sigma_0, \sigma_1, \ldots, \sigma_{\ell-1}) = (\sigma_n, \sigma_{n_k+1}, \ldots, \sigma_{n+\ell-1})$$

- (I) Algebraic irreducibility for each $k \in \mathbb{N}$, the characteristic polynomial of $M_{[k,\ell)}$ is irreducible for all sufficiently large ℓ
- (C) C-balance There is $C > 0$ such that $n = n(\ell)$ can be chosen such that $X_{\boldsymbol{\sigma}}^{(n+\ell)}$ has balance bounded by C
- (E) Generalized Left Eigenvector

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  cf. Furstenberg's condition

  There exists $h \in \mathbb{N}$ and a positive matrix B such that $M_{[\ell_k - h, \ell_k)} = B$ for all $k \in \mathbb{N}$

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$$\lim_{k\to\infty} \mathbf{v}^{(n_k)}/\|\mathbf{v}^{(n_k)}\| = \mathbf{v}$$

The PRICE to pay

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Theorem If (D, Σ, ν) is a Pisot S-adic shift such that there exists a cylinder of positive measure in D corresponding to a substitution with positive incidence matrix, then the property PRICE holds a.e.

Dendric subshifts

Extension graphs and dendric subshifts

We consider the set of factors $\mathcal{L}_X$ of a minimal subshift $X \subset A^{\mathbb{N}}$
Let $w \in \mathcal{L}_X$

$$\ell(w) = \{a \in A \mid aw \in \mathcal{L}_X\}$$

$$r(w) = \{a \in A \mid wa \in \mathcal{L}_X\}$$

$$e(w) = \{(a, b) \in A \times A \mid awb \in \mathcal{L}_X\}$$

The extension graph of the finite word w is the undirected graph $G(w)$ having

- $\ell(w)$ and $r(w)$ as vertices
- $e(w)$ as edges

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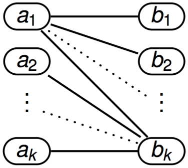

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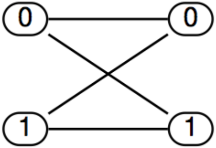
Definition We say that X is dendric if the graph $G(w)$ is a tree for any $w \in \mathcal{L}_X$

Tree = undirected, acyclic and connected graph
[B.,Berstel, Cecchi, De Felice, Delecroix, Dolce, Durand,Leroy, Petite, Perrin, Reutenauer, Rindone, etc.]

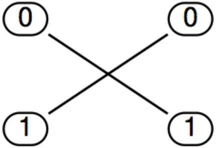
The Thue-Morse word is not a dendric word

$$\tau\colon 0 \mapsto 01,\ \ 1 \mapsto 10$$

$$u = \tau^{\infty}(0) = 01101001100101101001011001\cdots$$



$w = 01$

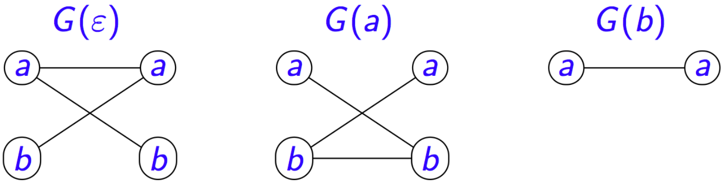


$w = 010$

The Fibonacci word is a dendric word

$$\sigma\colon a \mapsto ab,\ \ b \mapsto a$$

$$u = \sigma^\infty(a) = abaababaabaababaababaab\cdots$$



The factors of length 2 are aa, ab, ba

Examples of dendric words

A dendric word u on k letters has $(k-1)n+1$ factors of length n

- Sturmian words are dendric
- Arnoux-Rauzy words are dendric

$$l(w) = r(w) = 3$$

- Codings of interval exchanges are dendric

$$l(w) = r(w) = 2 \text{ for } w \text{ large enough}$$

Dendric subshifts are S-adic

Let $u \in A^{\mathbb{N}}$ be a uniformly recurrent dendric word over an alphabet of cardinality d

Theorem [B.,De Felice,Dolce,Leroy,Perrin,Reutenauer,Rindone]
Let w be a factor of u. The set of return words to w is a basis of the free group F_d.
The decoding of a uniformly recurrent dendric word u with respect to the return words of a given factor is again a dendric word. ☺

$\rightsquigarrow$ Dendric subshifts are S-adic, the substitutions are invertible

Theorem B.-Steiner-Thuswaldner-Yassawi] Let (X, T) be a minimal dendric shift. Consider a return word S-adic representation of (X, T). Then, the natural Bratteli-Vershik system associated with it is properly ordered and is topologically conjugate to (X, T). Its topological rank is bounded by the size of the alphabet of X.

Balancedess and dendric subshfits

Theorem [B.-Cecchi] Let (X, T) be a minimal dendric subshift. Then (X, T) is balanced on letters if and only if it is balanced on factors.
In particular, if (X, T) is balanced, then all the frequencies of factors are additive topological eigenvalues.

Example: Arnoux-Rauzy case

- The subshift generated by a primitive Arnoux-Rauzy substitution is balanced
- Let $(X_{\mathbf{i}}, T)$ be an Arnoux-Rauzy subshift on a three-letter alphabet with $\mathcal{S}_{AR}$-directive sequence $\mathbf{i} = (i_n)_{n\geq 0}$. If there exists some constant h such that we do not have $i_n = i_{n+1} = \cdots = i_{n+h}$ for any $n \geq 0$, then $(X_{\mathbf{i}}, T)$ is balanced [B.-Cassaigne-Steiner]

Image group of a dendric subshift

Let (X, S, μ) be a minimal and uniquely ergodic dendric subshift

$$I(X,S) = \left\{ \int f d\mu \, ; \, f \in C(X,\mathbb{Z}) \right\}$$

Theorem [B.-Cecchi-Dolce-Durand-Leroy-Perrin-Petite]

$$I(X,S) = \sum_{a \text{ letter in } \mathcal{A}} \mathbb{Z}\mu([a])$$

Proof

- For any $\alpha \in I(X,T) \cap (0,1)$, there exists a clopen set U such that $\alpha = \mu(U)$
- Extension graph The measure of any cylinder is in

$$\sum_{a \in \mathcal{A}} \mathbb{Z}\mu([a])$$

Frequencies of letters determine frequencies of factors

$\neq$ Thue-Morse $\mathbb{Z}[1/2]$ dyadic rationals