Dyadisc2 : Dynamiques et Analyse Discrète : analyse spectrale des sous-shifts

Spectral set conjecture on finite abelian groups

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Outline

Spectral set conjecture on $\mathbb{R}^d$

Spectral set conjecture on locally compact abelian groups

Spectral set conjecture on cyclic groups

Spectral sets in $\mathbb{R}^d$

- Let m be the Lebesgue measure in $\mathbb{R}^d$.
- Let $\Omega \subset \mathbb{R}^d$ be a Borel set with $0 < m(\Omega) < +\infty$.
- Definition. The set Ω is **spectral** if there exists a set $\Lambda \subset \mathbb{R}^d$ such that $E(\Lambda) = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis of $L^2(\Omega)$.
- The set Λ is called a **spectrum** of Ω and (Ω, Λ) is called a **spectral pair**.

Example:

- The d-dimensional cube $[0, 1]^d$ is spectral.
- $([0, 1] \cup [2, 3], \mathbb{Z} \cup (\mathbb{Z} + 1/4))$ is a spectral pair.
- The d-dimensional balls are NOT spectral: Iosevich–Katz–Pedersen (1999).
- The non-symmetric polytopes are NOT spectral: Kolountzakis (2000).

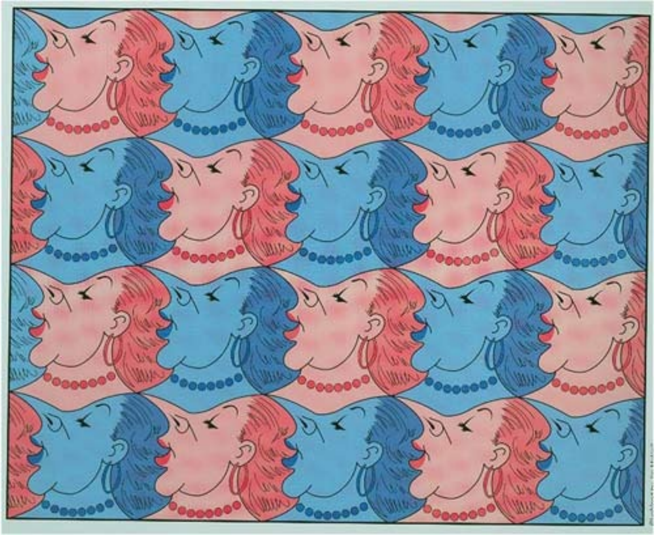
Tiles in $\mathbb{R}^d$

- Let $\Omega \subset \mathbb{R}^d$ be a Borel set with $0 < m(\Omega) < +\infty$.
- Definition. The set Ω **tiles** $\mathbb{R}^d$ by translations, if $\exists T \subset \mathbb{R}^d$ s. t. $\{\Omega + t : t \in T\}$ forms a partition a.e. of $\mathbb{R}^d$.
- The set T is called a **tiling complement** of Ω and (Ω, T) is called a **tiling pair**.

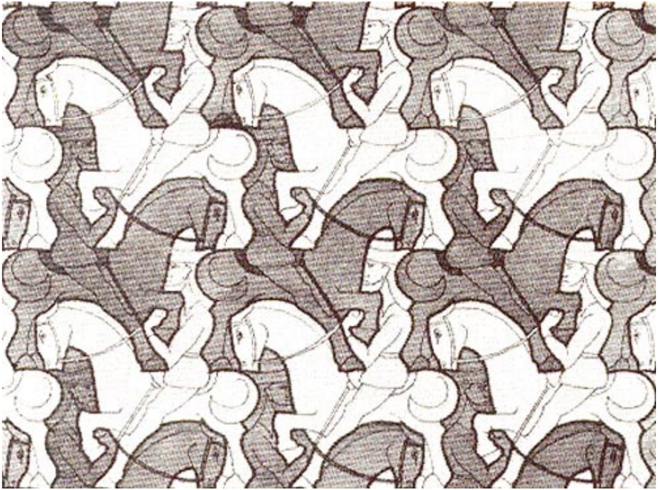
Example:

- The d-dimensional cube $[0, 1]^d$ is a tile
- $([0, 1] \cup [2, 3], 4\mathbb{Z} \cup (4\mathbb{Z} + 1))$ is a tiling pair.

Example



Counter-example



Spectral set conjecture in $\mathbb{R}^d$

Conjecture(Fuglede, 1974)

A Borel set Ω is a tile of $\mathbb{R}^d$ if and only if it is a spectral set.

- Segal's question: for which region Ω in $\mathbb{R}^d$, the partial differential operators in $C_c^\infty(\Omega)$ have commutative self-adjoint extensions in $L^2(\Omega)$?
- Conjecture holds if T or Λ is a lattice: Fuglede (1974).
- Spectral sets $\not\Rightarrow$ tiles for $d \geq 5$: Tao (2003).
- Conjecture fails for $d \geq 3$ (both direction): Matolsci(2005), Farkas-Gy(2006), Matolsci-Kolountzakis(2006), Farkas-Matolsci-Móra(2006):
- It remains open for $d = 1, 2$.

Partial positive results for $d = 1, 2$

- Union of two interval: Łaba (2001).
- Self-affine tiles: Bandt (1991), Kenyon (1992), Lagarias–Wang (1996, 1997), Lai–Lau–Rao (2013, 2017) . . . .
- Convex planar set: Iosevich, Katz and Tao (2003).
- . . . . . .

Tiles and spectral sets in LCA groups

$\mathbb{R}^d$	$\longleftrightarrow$	locally compact abelian (LCA) groups
Lebesgue measure	$\longleftrightarrow$	Haar measure
$\{e^{2i\pi\langle z,x\rangle} : z \in \mathbb{R}^d\}$	$\longleftrightarrow$	dual group $\widehat{G}$

- $(\{0,1\},\{0,2\})$ is a tile pair in $\mathbb{Z}/4\mathbb{Z}$.
- $(\{0,1\},\{0,2\})$ is a spectral pair in $\mathbb{Z}/4\mathbb{Z}$, here $\widehat{\mathbb{Z}/4\mathbb{Z}} \simeq \mathbb{Z}/4\mathbb{Z}$.

Spectral set conjecture in LCA groups

Conjecture

A Borel set Ω is a tile in G if and only if it is a spectral set.

- True in $\mathbb{Q}_p$: Fan–Fan–Liao–S (2016).
- Dutkay-Lai (2013):

(i) T-$S(G)$: Tile$\Rightarrow$Spectral direction holds in G.
T-$S(\mathbb{R}) \Leftrightarrow T$-$S(\mathbb{Z}) \Leftrightarrow T$-$S(\mathbb{Z}/m\mathbb{Z})$ for all $m \in \mathbb{N}$

(ii) S-$T(G)$: the Spectral$\Rightarrow$Tile direction holds in G.
S-$T(\mathbb{R}) \Rightarrow S$-$T(\mathbb{Z}) \Rightarrow S$-$T(\mathbb{Z}/m\mathbb{Z})$ for all $m \in \mathbb{N}$

Spectral set conjecture in finite groups

The conjecture is true for:

- $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, p prime: Iosevich-Mayeli-Pakianathan (2015).
- $\mathbb{Z}/p^n q\mathbb{Z}$, p, q distinct primes: Malikiosis-Kolountzakis (2016).
- T-$S(\mathbb{Z}/p^n q^m \mathbb{Z})$, p, q distinct primes: Łaba (2002).

Spectral sets $\not\Rightarrow$ tiles:

- $(\mathbb{Z}/3\mathbb{Z})^6$: Tao (2003).
- $(\mathbb{Z}/8\mathbb{Z})^3$: Kolountzakis–Matolcsi (2006).
- $(\mathbb{Z}/p\mathbb{Z})^4 \quad (p \equiv 1 \mod 4)$ et $(\mathbb{Z}/p\mathbb{Z})^5 \quad (p \equiv 3 \mod 4)$: Aten et al. (2017).

Tiles $\not\Rightarrow$ spectral sets:

- $(\mathbb{Z}/24\mathbb{Z})^3$: Farkas–Matolcsi–Mora (2006).

Tao's counterexample in $(\mathbb{Z}/3\mathbb{Z})^6$

- **Definition.** Hadamard matrix of order n: entries are n-th roots of unity and whose rows are mutually orthogonal.
- Example (Tao 2003): let $\omega = e^{\frac{2\pi i}{3}}$,

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega^2 & 1 & \omega^2 & \omega & 1 \end{pmatrix}$$

- Take $\Omega = \{e_1, e_2, \ldots, e_6\}$ to be the standard basis of $(\mathbb{Z}/3\mathbb{Z})^6$. Take $\Lambda = \{\xi_1, \xi_2, \ldots, \xi_6\}$ where $(e^{2\pi i(e_j \cdot \xi_k)})_{j=1}^6$ matches the k-th row of H.

Mask polynomial

- Let $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. Let $\omega_N = e^{2\pi i/N}$. Let $A \subset \mathbb{Z}_N$.
- Definition. The **mask polynomial** $A(X)$ is given by

$$\sum_{a \in A} X^a \in \mathbb{Z}[X]/(X^N - 1).$$

  - $\widehat{1}_A(n) = A(\omega_N^{-n}), \ \forall\, n \in \mathbb{Z}_N.$
- Definition. $\mathcal{Z}_A := \{n \in \mathbb{Z}_N : A(\omega_N^n) = 0\} = \{n \in \mathbb{Z}_N : \widehat{1}_A(n) = 0\}$.
  - $n \in \mathcal{Z}_A \Leftrightarrow nr \in \mathcal{Z}_A$ for any $r \in \mathbb{Z}_N^*$.

Spectral sets in cyclic groups

Let $A, B \subset \mathbb{Z}_N$. The following are equivalent.

(1) (A, B) is a spectral pair.

(2) $\sharp A = \sharp B$ and $(B - B) \setminus \{0\} \subset \mathcal{Z}_A$.

(3) $M = \left(e^{2\pi i \frac{ab}{N}}\right)_{b \in B, a \in A}$ is a complex Hadamard matrix.

Tiles in cyclic groups

Let $A, B \subset \mathbb{Z}_N$. The following statements are equivalent.

(1) (A, B) is a tiling pair.

(2) $A \oplus B = \mathbb{Z}_N$.

(3) $A(X)B(X) = 1 + X + X^2 + \cdots + X^{N-1} \mod X^N - 1$.

Coven–Meyerowitz property

- Let Φ_n be the cyclotomic polynomial of order n.
- Let $S := \{p^\alpha \mid N : p \text{ prime}\}$.
- Let $A \subset \mathbb{Z}_N$. $S_A := \{s \in S : \Phi_s(X) \mid A(X)\}$.
- Definition. **(T1)**: $\sharp A = \prod_{s \in S_A} \Phi_s(1)$.
- Definition. **(T2)**: $\Phi_{s_1 s_2 \ldots s_m}(X)$ divides $A(X)$ for every powers of distinct primes $s_1, s_2, \ldots, s_m \in S_A$.
- Let $d \mid N$.

$$d \in \mathcal{Z}_A \Longleftrightarrow \Phi_{N/d}(X) \mid A(X).$$

Property (T1): examples

Let $A \subset \mathbb{Z}_N$.

(1) $N = p^n$. (T1) is equivalent to

$$p^{\sharp(\mathcal{Z}_A \cap \{1, p, p^2, \dots, p^{n-1}\})} = \sharp A.$$

(2) $N = p^n q^m$. (T1) is equivalent to

$$p^{\sharp(\mathcal{Z}_A \cap \{q^m, pq^m, p^2 q^m, \dots, p^{n-1} q^m\})} \cdot q^{\sharp(\mathcal{Z}_A \cap \{p^n, p^n q, p^n q^2, \dots, p^n q^{m-1}\})} = \sharp A.$$

(3) $N = pqr$. (T1) holds if and only if

$$p^{\sharp(\mathcal{Z}_A \cap \{qr\})} \cdot q^{\sharp(\mathcal{Z}_A \cap \{pr\})} \cdot r^{\sharp(\mathcal{Z}_A \cap \{pq\})} = \sharp A.$$

Property (T2): examples

Let $A \subset \mathbb{Z}_N$.

(1) $N = p^n$. (T2) holds vacuously.

(2) $N = p^n q^m$. (T2) is equivalent to

$$p^a q^m, p^n q^b \in \mathcal{Z}_A \Rightarrow p^a q^b \in \mathcal{Z}_A.$$

(3) $N = pqr$. (T2) holds if and only if

$$pq, pr \in \mathcal{Z}_A \Rightarrow p \in \mathcal{Z}_A;$$

$$qr, pr \in \mathcal{Z}_A \Rightarrow r \in \mathcal{Z}_A;$$

$$pq, qr \in \mathcal{Z}_A \Rightarrow q \in \mathcal{Z}_A.$$

Spectral sets, tiles and (T1), (T2)

- (T1)+(T2) $\Rightarrow$ tiles for all $\mathbb{Z}_N$, $N \in \mathbb{N}$: Coven–Meyerowitz (1998).
- Tiles $\Rightarrow$ (T1) for all $\mathbb{Z}_N$, $N \in \mathbb{N}$: Coven–Meyerowitz (1998).
- Tiles $\Rightarrow$ (T1)+(T2) for $\mathbb{Z}_N$ with $N = p^n q^m$: Łaba (2002).
- Conjecture (Coven–Meyerowitz):

$$\text{(T1)+(T2)} \Leftrightarrow \text{tiles for all } \mathbb{Z}_N, N \in \mathbb{N}.$$

- (T1)+(T2) $\Rightarrow$ spectral sets for all $\mathbb{Z}_N$, $N \in \mathbb{N}$: Łaba (2002).
- Spectral sets $\Rightarrow$ (T1)+(T2) for $\mathbb{Z}_N$ with $N = p^n q$: Malikiosisa–Kolountzakis (2017).
- If for all $\mathbb{Z}_N$, $N \in \mathbb{N}$,

$$\text{Spectral sets } \Longleftrightarrow (T1) + (T2)$$

then S-$T(\mathbb{R})$: Dutkay-Lai (2013).

Main results

Theorem (S, 2018)

Let N be a square-free integer. Let $A \subset \mathbb{Z}_N$. Then A is a tile if and only if it satisfies (T1) and (T2).

Theorem (S, 2018)

Let p, q, r be distinct prime numbers. Let $A \subset \mathbb{Z}_{pqr}$. Then A is a spectral set if and only if it satisfies (T1) and (T2).

Corollary

The spectral set conjecture holds on $\mathbb{Z}_{pqr}$ with p, q, r distinct prime numbers.

Vanishing sums of roots of unity: I

For a given natural number N, what are the possible integers n for which there exist N-th roots of unity $e^{\frac{2\pi i\alpha_1}{N}}, e^{\frac{2\pi i\alpha_2}{N}}, \dots, e^{\frac{2\pi i\alpha_n}{N}}$ such that

$$e^{\frac{2\pi i\alpha_1}{N}} + e^{\frac{2\pi i\alpha_2}{N}} + \cdots + e^{\frac{2\pi i\alpha_n}{N}} = 0?$$

Theorem (Lam and Leung, 2000)

Let N be a positive integer. Let $A \subset \mathbb{Z}_N$. If $A(\omega_N) = 0$, then there exist $n_p \in \mathbb{N}$ for all p prime with $p \mid N$ such that $\sharp A = \sum_{p|N} n_p p$.

Vanishing sums of roots of unity: II

Proposition (Lam and Leung, 2000)

Let $n \mid N$ be such that N/n has at most two prime divisors, say p and q. If $A(\omega_N^n) = 0$, then

$$A(X^n) \equiv P(X^n)\Phi_p(X^{N/p}) + Q(X^n)\Phi_q(X^{N/q}) \mod X^N - 1,$$

where P and Q have nonnegative coefficients.

- The polynomial $A(X^n)$ is the mask polynomial of the multi-set $n \cdot A$.
- $\Phi_p(X^{N/p})$ is the mask polynomial of the subgroup $\frac{N}{p}\mathbb{Z}_N$. Its cosets are called p-cycles.
- The above proposition shows that $n \cdot A$ is the disjoint union of p-cycles and q-cycles.

N/n having three prime divisors

- Let $N = pqr$ with p, q, r distinct primes. Let A be a subset with mask polynomial

$$\begin{aligned} A(X) =&(X^{qr} + X^{2qr} + \cdots + X^{(p-1)qr})(X^{pr} + X^{2pr} + \cdots + X^{(q-1)pr}) \\ &+ (X^{pq} + X^{2pq} + \cdots + X^{(r-1)pq}), \end{aligned}$$

  we have $A(\omega_N) = 0$ but A cannot be expressed as a union of p-, q- and r-cycles

N/n having only one prime divisor

Proposition

Let $n \mid N$ be such that N/n has only one prime divisor, say p. If $A(\omega_N^n) = 0$, then

$$A(X^n) \equiv P(X^n)\Phi_p(X^{N/p}) \mod X^N - 1,$$

where P has nonnegative coefficients.

- $n \cdot A$ is the disjoint union of p-cycles.
- This is the key tool to prove that Spectral set conjecture holds in $\mathbb{Z}_{p^n}$: Łaba (2002), also Fan–Fan–S (2016).

Representation of $\mathbb{Z}_{pqr}$

$$\begin{array}{rcl} \mathbb{Z}_{pqr} & \longleftrightarrow & \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \\ x & \longleftrightarrow & (x_1, x_2, x_3) \end{array}$$

- $x_1 \equiv x \mod \mathbb{Z}_p$, $x_2 \equiv x \mod \mathbb{Z}_q$ and $x_3 \equiv x \mod \mathbb{Z}_r$.
- p-cycle: $\{(0, x_2, x_3), (1, x_2, x_3), \ldots, (p-1, x_2, x_3)\}$.

Sketch of proof

Lemma

Let (A, B) be a spectral pair in $\mathbb{Z}_{pqr}$. If $pq \notin \mathcal{Z}_B$, then there exist a subset $S \subset \mathbb{Z}_p \times \mathbb{Z}_q$ and a function $f : \mathbb{Z}_p \times \mathbb{Z}_q \to \mathbb{Z}_r$ such that

$$A = \{(x, y, f(x, y)) : (x, y) \in S\}.$$

Moreover, we have that $\sharp A \leq pq$ and that the equality holds if and only if $S = \mathbb{Z}_p \times \mathbb{Z}_q$.

Proof

If the set A has two elements (x, y, z) and (x, y, z') with $z \neq z'$, then we have

$$(x, y, z) - (x, y, z') = (0, 0, z - z') \in pq\mathbb{Z}_{pqr}^*.$$

It follows that $pq \in \mathcal{Z}_B$, which is a contradiction.

Sketch of proof

- We decompose the proof into three cases:
  (1) $\sharp(\mathcal{Z}_A \cap \{pq, pr, qr\}) = 2$;
  (2) $\sharp(\mathcal{Z}_A \cap \{pq, pr, qr\}) = 1$;
  (3) $\sharp(\mathcal{Z}_A \cap \{pq, pr, qr\}) = 0$.

Proof of (1)

WLOG, $qr, pr \in \mathcal{Z}_A$ and $pq \notin \mathcal{Z}_A$. It follows that $pq \mid \sharp A = \sharp B$. We thus have $pq \notin \mathcal{Z}_B$ and $\sharp A = pq$. (T1) holds. There exists a function $f : \mathbb{Z}_p \times \mathbb{Z}_q \to \mathbb{Z}_r$ such that

$$A = \{(x, y, f(x, y)) : (x, y) \in \mathbb{Z}_p \times \mathbb{Z}_q\}.$$

We obtain that $r \in \mathcal{Z}$ and thus A satisfies the (T2).

Sketch of proof

Proof of (3)

Let B be a spectrum of A.
Step 1: $\sharp(\mathcal{Z}_B \cap \{pq, pr, qr\}) = 0$.
Step 2: $\sharp A = \sharp B < \min\{p, q, r\}$. This implies that $\sharp A = \sharp B = 1$.

Proof of (2)

Let B be a spectrum of A. WLOG, $qr \in \mathcal{Z}$ and $pq, pr \notin \mathcal{Z}$. (T2) holds vacuously.
Step 1: $\sharp(\mathcal{Z}_B \cap \{pq, pr, qr\}) = 1$.
Step 2: $\sharp A = \sharp B = p$. It follows that (T1) holds.

Open questions

Dimension one:

- $\mathbb{Z}/p^n q\mathbb{Z}$, p, q distinct primes: Malikiosis–Kolountzakis (2016).
- $\mathbb{Z}/p^n q^6\mathbb{Z}$, p, q distinct primes: Malikiosis (in progress).
- T-$S(\mathbb{Z}/p^n q^m\mathbb{Z})$, p, q distinct primes: Łaba (2002).
- $\mathbb{Z}/p^n q^m\mathbb{Z}$? In general, $\mathbb{Z}/m\mathbb{Z}$?

Higher dimension:

- $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, p prime: Iosevich–Mayeli–Pakianathan (2015).
- $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, p prime: S (in progress).
- $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$? In general, $\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$?
- $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$?

Thanks for your attention!