Dimension group of dendric subshifts

Paulina CECCHI B.
(Joint work with Valérie Berthé, F. Dolce, F. Durand, J. Leroy, D. Perrin and S. Petite)

Institute de Recherche en Informatique Fondamentale
Université Paris Diderot - Paris 7

Departamento de Matemática y Ciencia de la Computación
Facultad de Ciencia. Universidad de Santiago de Chile

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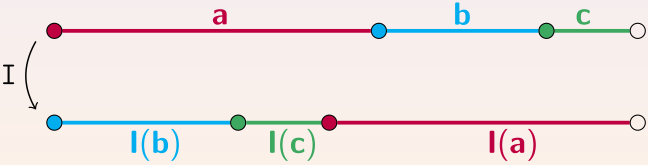

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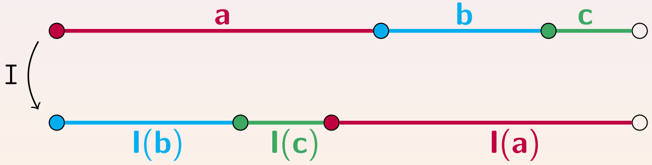

- How can we compare this two systems?

Motivation

- In the Tribonacci subshift, there exists a unique infinte word $x_G = x_0 x_1 x_2 \cdots$ with three different pasts: the words

$$y_1 = p_1(x_G) \cdot x_G$$
$$y_2 = p_2(x_G) \cdot x_G$$
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- In the three-interval exchange there are two infinite words x_G^1, x_G^2 with two different pasts $\Rightarrow$ two asymptotic pairs.
- Since proper asymptotic pairs are preserved under conjugacy, both subshifts are not conjugate.
- But they are orbit equivalent, they are even strong orbit equivalent.

Outline

- Orbit equivalence and Dimension group of a subshift.
- Dendric subshifts.
- Dimension group of dendric subshifts.

Orbit equivalence and Dimension group

Orbit equivalence

- Two (topological) dynamical systems (X, T) and (Y, S) are (topological) orbit equivalent if there is a homeomorphism $h : X \to Y$ such that for all $x \in X$

$$h(\{T^n(x) : n \in \mathbb{Z}\}) = \{S^n(h(x)) : n \in \mathbb{Z}\},$$

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- If (X, T) and (Y, S) are minimal, there exist a maps $n \colon X \to \mathbb{Z}$ (the cocycle map) such that, for all $x \in X$,

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- (X, T) and (Y, S) are strong orbit equivalent if n has at most one point of discontinuity.

Dimension group

- Consider $C(X, \mathbb{Z})$ the set of continuous functions from X to $\mathbb{Z}$, $\beta : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ given by

$$\beta f = f \circ T - f.$$

  Images of β are called coboundaries.

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- Define $H(X,T) := C(X,\mathbb{Z})/\beta C(X,\mathbb{Z})$.
- Is is a partially ordered abelian group, whose positive cone is given by

$$H(X,T)^+ = \{[f] \in H(X,\mathbb{Z}) : f \in C(X,\mathbb{N})\}.$$

  $([f] \geq [g] \Leftrightarrow [f]-[g] \in H(X,T)^+)$.

Dimension group

- The class $[1]$ of the constant function 1 is an order unit of $H(X,T)$: for every $[f] \in H(X,T)$, there exists $n \in \mathbb{N}$ such that

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$$K^0(X,T) = (H(X,T), H(X,T)^+, [1])$$
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Theorem (Giordano–Putnam–Skau '95)

(X, T) and (Y, S) are strong orbit equivalent if an only if
$$K^0(X, T) \cong K^0(Y, S)$$
(as ordered group with unit).

Dimension group

Example. The dimension groups associated to the Tribonacci shift and the three-interval exchange with frequencies $(\alpha, \alpha^2, \alpha^3)$ are both

$$\left(\mathbb{Z}^3,\, \{\mathbf{x} \in \mathbb{Z}^3 \mid \langle \mathbf{x}, \mathbf{f} \rangle > 0\} \cup \{0\},\, \mathbf{1}\right).$$

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- So the two subshifts are strong orbit equivalent.

Dendric subshifts

Dendric subshifts

- Recall that for any factor w in the language $\mathcal{L}_X$ of (X, T), the extensions of w are the following sets,

$$\begin{aligned} L(w) &= \{a \in \mathcal{A} \mid aw \in \mathcal{L}_X\} \\ R(w) &= \{a \in \mathcal{A} \mid wa \in \mathcal{L}_X\} \\ B(w) &= \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}_X\}. \end{aligned}$$

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- Left special factor: $|L(w)| \geq 2$
  Right special factor: $|R(w)| \geq 2$
  Bispecial factor: $|L(w)|, |R(w)| \geq 2$.

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- The extension graph $\mathcal{E}(w)$ of w is the undirected bipartite graph whose set of vertices is the disjoint union of $L(w)$ and $R(w)$ and whose edges are the pairs $(a, b) \in B(w)$.

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  - Consider the Thue-Morse word in $\{a, b\}$ given by

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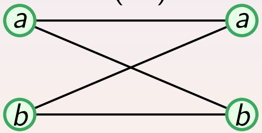
  - The extension graph of ab is

$\mathcal{E}(ab)$



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- We focus on minimal dendric subshifts.

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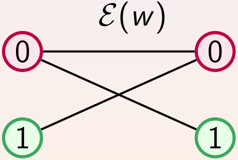

Dendric subshifts: Arnoux–Rauzy words

- Consider the alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$, $x \in \mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$ is an Arnoux-Rauzy word if it is uniformly recurrent and for each $n \in \mathbb{N}$ there exists exactly

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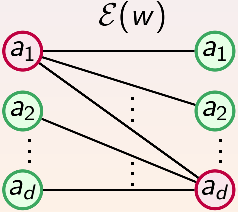

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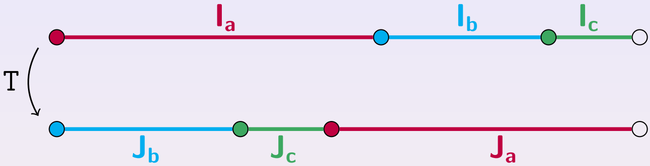
- The interval exchange transformation relative to $(I_a)_{a\in\mathcal{A}}$ is the map $T : [0,1) \to [0,1)$ given by

$$T(z) = z + (\nu_a - \mu_a) \quad if \quad z \in I_a.$$

Dendric subshifts: Interval exchanges

- This corresponds to



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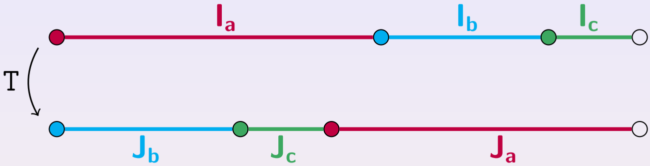

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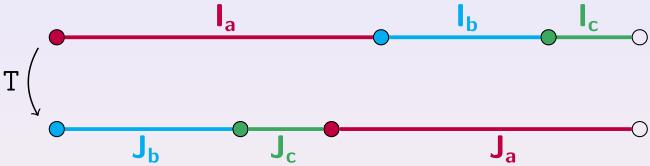

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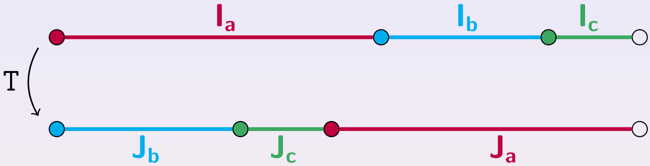

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- [Keane '75] Regular interval exchanges are minimal.

- [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone '14] Regular interval exchanges generate dendric subshifts.

Dendric subshifts: Interval exchanges

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AR on d letters.

Exchange of d intervals.

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Dendric subshifts

Dimension group of dendric subshifts

Theorem (Berthé, C., Durand, Dolce, Leroy, Perrin, Petite '18)

Let (X, T) be a minimal dendric subshift on a d-letter alphabet. Let $\mathcal{M}(X, T)$ stand for its set of invariant measures. Then, $K^0(X, T)$ is isomorphic to

$$\left(\mathbb{Z}^d, \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \mathbf{f}_\mu \rangle > 0 \textit{ for all } \mu \in \mathcal{M}(X, T)\} \cup \{\mathbf{0}\}, \mathbf{1}\right),$$

where $\mathbf{f}_\mu \in \mathbb{R}^d$ is the letter frequency vector.

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Corollary

Two minimal dendric subshifts over the same alphabet are strong orbit equivalent if and only they have the same additive group of letter frequencies.

Return words

- Let $w \in \mathcal{L}_X$. A left return word to w is a factor $u \in \mathcal{L}_X$ such that $uw \in \mathcal{L}_X$, w is a prefix of uw and uw contains exactly two occurrences of w.

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produced by the substitution $\varphi : a \mapsto ab; b \mapsto a$.
The return words to the prefixes of x_F are:

$$\text{return words to } a = \{a, ab\}.$$
$$\text{return words to } ab = \{ab, aba\}.$$
$$\text{return words to } aba = \{ab, aba\}.$$
$$\vdots$$

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Theorem (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Ridone '15)

Let (X, T) a minimal dendric subshift on the alphabet $\mathcal{A}$. Then for any $w \in \mathcal{L}_X$, the set $\mathcal{R}_X(w)$ is a basis of the free group on $\mathcal{A}$.

Partitions in towers

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- Denote by $\{v_1(n), \cdots, v_d(n)\}$ the set of d return words to $x_{[0,n)}$.

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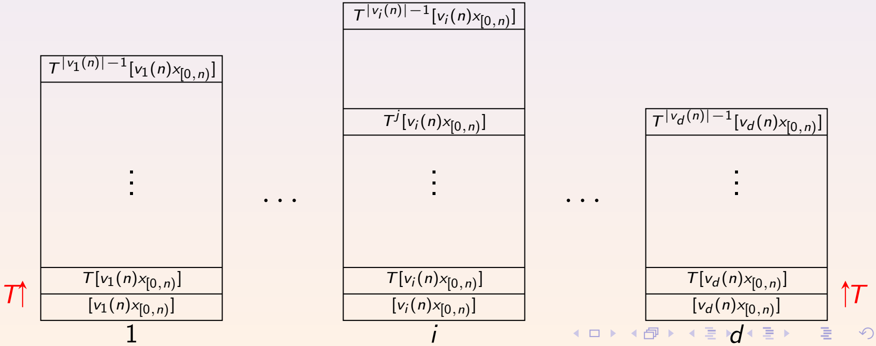

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- Since $\{v_1(n), \cdots, v_d(n)\}$ is a basis of the free group F_d for all n,

$$M_n \in GL_d(\mathbb{Z}),$$

(every $v_i(n)$ admits a unique decomposition in terms of $v_k(n+1)$'s and their inverses).

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$$\varphi[(x_n)_{n\geq 0}] \longmapsto (M_k \cdot M_{k-1} \cdots M_1 \cdot M_0)^{-1}(x_k)$$

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- Note we have added the partition $\mathcal{P}_0 = \{[a] : a \in \mathcal{A}\}$. The matrix M_0 whose coefficients are

$$M_0(k,i) = |v_k(1)|_a$$

is invertible.

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- So that
$$(\mathbb{Z}^d, ?, \mathbf{1}) \xrightarrow{\varphi^{-1}} (G_{\mathfrak{S}}, G_{\mathfrak{S}}^+, \mathbf{1}_{\mathfrak{S}}) \xrightarrow{\pi_{\mathfrak{S}}} (H(X, T), H(X, T)^+, \mathbf{1})$$

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- Fact:

$$H(X,T)^+ = \pi_{\mathfrak{S}} \circ \varphi^{-1}(\{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \mathbf{f}_\mu \rangle > 0 \text{ for all } \mu \in \mathcal{M}(X,T)\} \cup \{0\}).$$

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- To prove the fact, we use two fundamental tools,
  (1) [Effros '81] The positive cone $H(X,T)^+$ is completely determined by invariant measures,

$$H(X,T)^+ = \left\{\pi(f) \in H(X,T) : \int f d\mu > 0 \forall \mu \in \mathcal{M}(X,T)\right\} \cup \{0_{H(X,T)}\}$$

  (2) Matrices M_n are invertible.